Th: Let S be a non-empty subset of $\mathbb{R}$ bounded below. Then S has an infimum

Proof: Let l_0 be a lower bound of S.

Let $T = \{ l \in \mathbb{R} : l$ is a lower bound of $S \}$

Then clearly $l_0 \in T$. So T is non-empty.

Moreove, $x \in T$ and $s \in S \Rightarrow x \leq s$.

This shows that T is bounded above.

Thus T is a non-empty subset of $\mathbb{R}$, bounded above

By the suprimum property of $\mathbb{R}$, T has a suprimum. Let $\sup T = L$.

Then (i) $t \leq L, \forall t \in T$

(ii) since ~~$\forall$~~ $s \in S$ is an upper bount of T and $L = \sup T$.

$\therefore L \leq s \;\forall s \in S$.

L is ~~to~~ a lower bound of S and.

$L \geq$ any lower bond of S.

$\therefore$ consequently $L = \inf S$.

Therefor S has an infimum.

Th: Let S be a non-empty subset of $\mathbb{R}$, bounded above. An upper bound u of S is the supriemum of S iff each $\varepsilon > 0$ $\exists$ an element s in S such that $u - \varepsilon < s \leq u$.

Proof: Let $u = \sup S$. Let us choose $\varepsilon > 0$.

Then $u - \varepsilon$ is not an upper bound of S.

$\therefore \exists$ a leas one element $s \in S$ s.t

$s > u - \varepsilon$

since $u = \sup S$ and $s \in S$, we have ~~s~~ $\leq u$

$\therefore u - \varepsilon < s \leq u$

Conversely, let u be an upper bound of S. and for any $\varepsilon > 0$ $\exists$ $s \in S$ s.t $\boxed{u-\varepsilon < s \leq u}$, we prove that u is the least upper bound of S.

If possible, let u_0 be an upper bound of S such $u_0 < u$. Let $\varepsilon = \frac{1}{2}(u-u_0) > 0$ $\left[\because u_0 < u \Rightarrow u - u_0 > 0\right]$

~~Then $u-\varepsilon = u - \frac{1}{2}(u-u_0) = \frac{1}{2}u + u_0$~~

Now $2\varepsilon = u - u_0$

$\therefore$ $u - \varepsilon = \varepsilon + u_0$.

So by the given condition, $\exists$ $s' \in S$ s.t

$\varepsilon + u_0 = u - \varepsilon < s' \leq u$.

$\Rightarrow$ $\varepsilon + u_0 < s' \leq u$.

$\Rightarrow$ $s' > \varepsilon + u_0 > u_0$

$\Rightarrow$ u_0 can not be an upper bound of S.

Hence U is the least upper bound of S.

Th: Let S be a non-empty subset of $\mathbb{R}$, and bounded below. An lower bound l of S is infimum of S iff for any $\varepsilon > 0$ $\exists$ $s \in S$ s.t

$l \leq s < l + \varepsilon$.

Ex! prove that $\mathbb{N}$ is not bounded above.

$\Rightarrow$ Clearly $1 \in \mathbb{N}$, so $\mathbb{N} \neq \phi$. So $\mathbb{N}$ is a non-empty subset of $\mathbb{R}$.

Let $\mathbb{N}$ is bounded above.
Then $\mathbb{N}$ become a non-empty bounded above subset of $\mathbb{R}$.
By suprimum property $\mathbb{N}$ has suprimum. Let $u = \sup \mathbb{N}$.

Then (i) $\forall x \in \mathbb{N}, x \leq u$

(ii) for any $\varepsilon > 0$ $\exists$ $y \in \mathbb{N}$ s.t $u - \varepsilon < y \leq u$.

Let $\varepsilon = 1$, $\exists$ $k \in \mathbb{N}$ s.t

$u - 1 < k \leq u$

$\Rightarrow u - 1 < k$

$\Rightarrow \underline{u < k+1}$

So $k \in \mathbb{N}$, $k+1 \in \mathbb{N}$ and u is a upper bound. we get a $k+1 \in \mathbb{N}$ s.t $u < k+1$ which is contradiction, that u isn't upper bound of $\mathbb{N}$.

So $\mathbb{N}$ isn't bounded above.