Date: 14.09.20

System of Linear Algebraic Equation

Iterative method

Let us consider a system of n-linear algebric equation in n unknown —

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots \quad \cdots \quad & \cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}\right\} \quad (1)$$

where a_{ij} $\begin{pmatrix} i = 1(1)n \\ j = 1(1)n \end{pmatrix}$ are the coefficients, b_i $(i=1(1)n)$ are the known values and x_i, $i=1(1)n$ are the unknowns to be determined.

The system can be written as matrix form
$AX = b$ where $A = (a_{ij})_{n\times n}$ $X = (x_i)_{1\times n}$ $b = (b_i)_{1\times n}$.

A general linear iterative method for the solution of the system of eqn (1) may be define in the form

$$x^{(k+1)} = Hx^{(k)} + C, \quad k = 0, 1, 2 \cdots \longrightarrow (2)$$

where $x^{(k+1)}$ an $x^{(k)}$ are the approximations for x at $(k+1)$th and kth iterations, respectively. H is called the iteration matrix depending on A and C is a column vector. In the limiting case when $k \to \infty$, $x^{(k)}$ converges to the exact solution $x = A^{-1}b \longrightarrow (3)$

And the iteration equation (2) becomes, by substituti from (3) $\quad A^{-1}b = HA^{-1}b + C$

$$\Rightarrow C = (I - H)A^{-1}b$$

We now determine the iteration matrix H and the column vector C for a few well known iteration methods —

Jacobi Iteration method

We consider the system of eqn (1). Then a_{ii} in (1) $i=1(1)n$ becomes pivot elements.

Then (1) may be written as

$$a_{11}x_1 = b_1 - (a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n)$$
$$a_{22}x_2 = b_2 - (a_{21}x_1 + a_{23}x_3 + \cdots + a_{2n}x_n).$$
$$\vdots$$
$$a_{nn}x_n = b_n - (a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{n,n-1}x_{n-1})$$

The Jacobi iteration method or Gauss Jacobi iteration method may now be define as.

$$x_1^{(k+1)} = -\frac{1}{a_{11}}\left(a_{12}x_2^{(k)} + a_{13}x_3^{(k)} + \cdots + a_{1n}x_n^{(k)} - b_1\right)$$
$$x_2^{(k+1)} = -\frac{1}{a_{22}}\left(a_{21}x_1^{(k)} + a_{23}x_3^{(k)} + \cdots + a_{2n}x_n^{(k)} - b_2\right)$$
$$\cdots \quad \cdots$$
$$x_n^{(k+1)} = -\frac{1}{a_{nn}}\left(a_{n1}x_1^{(k)} + a_{n2}x_2^{(k)} + \cdots + a_{n,n-1}x_{n-1}^{(k)} - b_n\right). \quad \longrightarrow (4)$$

Since, we replace the complete vector $x^{(k)}$ in the right side of (4) at the end of each iteration, this method is also called the method of simultaneous displacement.

In matrix form, the method can be written as.

$$x^{(k+1)} = -D^{-1}(L+U)x^{(k)} + D^{-1}b.$$
$$= Hx^{(k)} + c$$

where $H = -D^{-1}(L+U)$ $c = D^{-1}b$.

where L and U are respectively lower and upper triangular matrices with zero diagonal entries, D is the diagonal matrix such that $A = L+D+U$

Alternatively $x^{(k+1)} = x^{(k)} - [I + D^{-1}(L+U)]x^{(k)} + D^{-1}b.$

$$= x^{(k)} - D^{-1}[D+L+U]x^{(k)} + D^{-1}b.$$
$$= x^{(k)} + D^{-1}[b - Ax^{(k)}]$$

or $v^{(k)} = x^{(k+1)} - x^{(k)}$

is the error in the approximation and

$r^{(k)} = b - Ax^{(k)}$ is the residual vector.

we may rewrite the above eqn is

$$D v^{(k)} = r^{(k)}.$$

we solve for $v^{(k)}$ and find $x^{(k+1)} = x^{(k)} + v^{(k)}$.

These equations describe the Jacobi iteration method in an error format.

Show that the LU decomposition method fails to solve the system of eqn

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & 5 \\ 3 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix}$$

Exact Solution is $x_1 = 1$
$x_2 = 0$
$x_3 = -1$