D'Alembert's Ratio Test

Statement:

1. If $\sum_{n=1}^{\infty} u_n$ be a series of positive terms such that $\lim \frac{u_{n+1}}{u_n} < 1$, then the series $\sum u_n$ converges.

2. If $\underline{\lim} \frac{u_{n+1}}{u_n} > 1$, then the series $\sum u_n$ diverges.

Proof of D'Alembert's Ratio Test

First Part: let $\lim \frac{u_{n+1}}{u_n} = l \ (<1)$.

It is always possible to choose an $\varepsilon (>0)$, such that $0 < l < l+\varepsilon < 1$.

Let $l+\varepsilon = k \ (<1)$, then $\exists \ m \in N$ such that $\frac{u_{n+1}}{u_n} < k \ (<1)$, $\forall n \geq m$.

$\Rightarrow u_{n+1} < k . u_n \ , \ \forall n \geq m$.

Thus for $n \geq m$, we have

$u_n < k u_{n-1} < k . (k u_{n-2}) = k^2 u_{n-2} < \dots < k^{n-m} u_m$

Hence $u_n < \left(\frac{u_m}{k^m}\right) . k^n$, for $n \geq m$.

But $\sum k^n$ is a convergent series, as $0 < k < 1$. Therefore, the series $\sum u_n$ converges, by Comparison Test

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1 2 3 4 5 6
7 8 9 10 11 12 13
14 15 16 17 18 19 20
21 22 23 24 25 26 27
28 29 30 31
M T W T F S S

Second Part:– Let $\lim \frac{u_{n+1}}{u_n} = l \; (>1)$.

It is always possible to choose an $\varepsilon \; (>0)$, such that $1 < l-\varepsilon < l$.

Then $\exists \; m \in N$, such that $\frac{u_{n+1}}{u_n} > \lambda, \; \forall n \geq m$ $(\lambda = l - \varepsilon)$.

Proceeding as before, for $n > m$, we have $u_n > \left(\frac{u_m}{\lambda^m}\right) . \lambda^n$ for $n \geq m$

Hence $\frac{u_m}{\lambda^m} > 0$, a constant and $\sum \lambda^n$ is a divergent series, as $\lambda \; (= l - \varepsilon) > 1$.

Therefore, the series $\sum u_n$ diverges, by Comparison Test.

Thus the D' Alembert's Ratio Test is proved for both the cases.

30	31					1
2	3	4	5	6	7	8
9	10	11	12	13	14	15
16	17	18	19	20	21	22
23	24	25	26	27	28	29
S	M	T	W	T	F	S

JULY '17