Date: 2.11.20

Interpolation and Approximation

[1] Newton's Forward interpolation :

we assume that we are given an interval $[a,b]$ and a function $f(x)$ which is continuous on $[a,b]$. Further we assume that we have $(n+1)$ distinct equispaced points $a \le x_0 < x_1 < x_2 < \cdots < x_n \le b$ of $[a,b]$. such that $x_r = x_0 + rh$ $(r = 0,1,2 \cdots n)$. where h is the length of each space. and corresponding entries are $f(x_0) = y_0$, $f(x_1) = y_1, \cdots f(x_r) = y_r, \cdots f(x_n) = y_n$.

Now $x_n - x_r = x_0 + nh - x_0 - rh = (n-r)h$.

we seek to find the polynomial $P(x) = a_0 + a_1 x + \cdots + \cdots + a_n x^n$ which satisfing the interpolating condition $P(x_i) = f(x_i)$ $i = 0,1,\cdots n$.

For Newton's Forward interpolation, we taken $P(x)$ as

$$P(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \cdots + a_r(x-x_0)(x-x_1)\cdots(x-x_{r-1}) + \cdots + a_n(x-x_0)(x-x_1)\cdots(x-x_{n-1}).$$

If we know $a_0, a_1, \cdots a_n$ then we gate the poln $P(x)$

To determine the value of $a_0, a_1 \cdots a_n$ we use the interpolating condition i.e $P(x_i) = f(x_i)$ $i = 0,1,\cdots n$

Now $P(x_0) = f(x_0)$

$\Rightarrow a_0 + 0 = f(x_0) = y_0$

$P(x_1) = f(x_1)$

$\Rightarrow a_0 + a_1(x_1 - x_0) = f(x_1)$

$\Rightarrow a_1(x_1 - x_0) = f(x_1) - f(x_0)$

$\Rightarrow a_1(x_1 - x_0) = f(x_1) - f(x_0)$

$\Rightarrow a_1 = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0} = \dfrac{\Delta f(x_0)}{h}$

$P(x_2) = f(x_2)$

$\Rightarrow a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) = f(x_2)$

$\Rightarrow a_2(x_2 - x_0)(x_2 - x_1) = f(x_2) - f(x_0) - \dfrac{f(x_1) - f(x_0)}{(x_1 - x_0)}(x_2 - x_0)$

$\Rightarrow a_2 = \dfrac{f(x_2) - f(x_0)}{(x_2 - x_0)(x_2 - x_1)} - \dfrac{f(x_1) - f(x_0)}{(x_1 - x_0)(x_2 - x_1)}$

$$a_2 = \frac{f(x_2) - \cancel{f(x_1) + f(x_0)}) - f(x_0)}{2h \cdot h} - \frac{f(x_1) - f(x_0)}{h \cdot 2h \cdot h}$$

$$= \frac{f(x_2) - f(x_0) - 2f(x_1) + 2f(x_0)}{2h^2}$$

$$= \frac{f(x_2) - 2f(x_1) + f(x_0)}{2h^2}$$

$$= \frac{f(x_2) - f(x_1) - f(x_1) + f(x_0)}{2h^2}$$

$$= \frac{\Delta f(x_1) - \{f(x_1) - f(x_0)\}}{2h^2}$$

$$= \frac{\Delta f(x_1) - \Delta f(x_0)}{2h^2}$$

$$= \frac{\Delta^2 f(x_0)}{2h^2}$$

Now $p(x_3) = f(x_3)$

$$\Rightarrow a_0 + a_1(x_3 - x_0) + a_2(x_3 - x_0)(x_3 - x_1) + a_3(x_3 - x_0)(x_3 - x_1)(x_3 - x_2) = f(x_3)$$

$$\Rightarrow a_3(x_3 - x_0)(x_3 - x_1)(x_3 - x_2) = \cancel{f(x_3) - f(x_0) - \{f(x_1) - f(x_0)\}(x_3 - x_0)\}} \cancel{- \{f(x_2) - 2f(x_1) + f(x_0)\}}$$

$$= f(x_3) - f(x_0) - \frac{\Delta f(x_0)}{h} 3h - \frac{\Delta^2 f(x_0)}{2h^2} \cdot 3h \cdot 2h$$

$$\Rightarrow a_3 = \frac{f(x_3) - f(x_0) - 3\{f(x_1) - f(x_0)\} - 3\{f(x_2) - 2f(x_1) + f(x_0)\}}{3h \cdot 2h \cdot h}$$

$$= \frac{f(x_3) - 3f(x_2) + 3f(x_1) - f(x_0)}{3!\, h^3}$$

$$= \frac{f(x_3) - f(x_0) - 3\Delta f(x_1)}{3!\, h^3}$$

$$= \frac{\Delta^3 f(x_0)}{3!\, h^3}$$

$\because \Delta f(x_0) = f(x_1) - f(x_0)$

$\Delta^2 f(x_0) = \Delta f(x_1) - f(x_0)$

$\Delta^3 f(x_0) = \Delta^2 f(x_1) - \Delta f(x_0)$

similarly $a_i = \dfrac{\Delta^i f(x_0)}{i!\, h^i}$

$$\therefore p(x) \simeq f(x_0) + (x - x_0)\frac{\Delta f(x_0)}{h} + (x - x_0)(x - x_1)\frac{\Delta^2 f(x_0)}{2!\, h^2} + (x - x_0)(x - x_1)(x - x_2)\frac{\Delta^3 f(x_0)}{3!\, h^3} + \cdots + (x - x_0)(x - x_1)\cdots(x - x_{n-1})\frac{\Delta^n f(x_0)}{n!\, h^n}$$

we now transfer the last formula to a more convenient and useful form by introducing a dimensionless quantity u, called phase, given by

$$u = \frac{x - x_0}{h} \quad \text{or} \quad x = x_0 + hu$$

Also $\quad x_r = x_0 + rh.$

$$x - x_r = (u - r)h$$

$$P(x) \simeq f(x_0) + u \cdot \Delta f(x_0) + u(u-1)\frac{\Delta^2 f(x_0)}{2!} + u(u-1)(u-2)\frac{\Delta^3 f(x_0)}{3!}$$
$$+ \cdots + u(u-1)(u-2)\cdots(u-\overline{n-1})\frac{\Delta^n f(x_0)}{n!}$$

this formula is called Newton's Forward interpolation formula.