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Interpolation

we know that if f be a function, continuous on $[a,b]$ and $a \le x_0 \le x_1 < x_2 < \cdots < x_n \le b$ be $n+1$ points on $[a,b]$, and f is know at this points, if we want to find the pol^n of degree $\le n$, i.e. $P(x) = a_0 + a_1 x + \cdots + a_n x^n$ s.t $f(x_i) = P(x_i)$ $i = 0, 1, \ldots n$. and this pol^n is called interpolating pol^n.

For Lagrange's interpolation the Lagrange interpolating pol^n of degree n is $P(x) = \sum_{i=0}^{n} l_i(x) f(x_i)$

where
$$l_i(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}$$
$i = 0, 1, \ldots n$.

which is can be written as $l_i(x) = \dfrac{w(x)}{(x-x_i)\,w'(x_i)}$

$w(x) = (x-x_0)(x-x_1)\cdots(x-x_n)$.

Now we calculate the truncation error in the Lagrange interpolation.

Let $P(x)$ be a pol^n coincides with the function $f(x)$ at $x_0, x_1, \ldots x_n$ and it deviates at all other points in the interval $[a,b]$. This deviation is called truction error and which is written as
$$E_n(f;x) = f(x) - P(x).$$

Since $f(x_i) = P(x_i)$ $i = 0, 1, \ldots n$, so $E_n(f;x) = 0$ at $x = x_i$ $i = 0, 1, \ldots n$. $\forall x \in [a,b]$, $x \ne x_i$.

we define a function
$$g(t) = f(t) - P(t) - [f(x) - P(x)] \frac{(t-x_0)(t-x_1)\cdots(t-x_n)}{(x-x_0)(x-x_1)\cdots(x-x_n)}.$$

we observe that $g(t) = 0$ at $t = x$ and $t = x_i$, $i = 0, 1, \ldots n$.

Applying the Rolle's theorem repeatedly for $g(t), g'(t), \ldots g^{(n)}(t)$, we obtain $g^{(n+1)}(\xi) = 0$. where ξ is some point such that
$$\min(x_0, x_1, \ldots x_n, x) < \xi < \max(x_0, x_1, \ldots x_n, x)$$

Differentiating $g(t)$, $n+1$ times with respect to t, we get

$$g^{(n+1)} = f^{(n+1)}(t) - \frac{(n+1)!\,[f(x) - P(x)]}{(x-x_0)(x-x_1)\cdots(x-x_n)}$$

setting $g^{(n+1)}(\xi) = 0$

$$\Rightarrow f^{(n+1)}(\xi) - \frac{(n+1)!\,[f(x)-P(x)]}{(x-x_0)(x-x_1)\cdots(x-x_n)} = 0$$

$$\Rightarrow \frac{(n+1)!\,[f(x)-P(x)]}{(x-x_0)(x-x_1)\cdots(x-x_n)} = f^{(n+1)}(\xi)$$

$$\Rightarrow f(x) - P(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n).$$

$$\Rightarrow E_n(f;x) = \frac{f^{(n+1)}(\xi)\cdot w(x)}{(n+1)!}$$

Ex: Given that $f(0)=1$, $f(1)=3$, $f(3)=55$, find the unique polynomial of degree 2 or less, which fits the given data. Find the bounds on the error.

$\Rightarrow$ We have $x_0 = 0$, $f(0)=1$

$x_1 = 1$, $f(1) = 3$

$x_2 = 3$, $f(3) = 55$

The Lagrange quadratic interpolating polynomial is given by

$$P_2(x) = l_0(x)f(x_0) + l_1(x)f(x_1) + l_2(x)f(x_2) \rightarrow (1)$$

Now $l_i(x)$ is called Lagrange fundamental polynomial.

$$\therefore l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-1)(x-3)}{(0-1)(0-3)} = \frac{1}{3}(x^2-4x+3)$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-0)(x-3)}{(1-0)(1-3)} = -\frac{1}{2}(x^2-3x)$$

$$= \frac{1}{2}(3x-x^2)$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-0)(x-1)}{(3-0)(3-1)} = \frac{1}{6}(x^2-x)$$

$$\therefore P_2(x) = \frac{1}{3}(x^2-4x+3)\times 1 + \frac{1}{2}(3x-x^2)\times 3 + \frac{1}{6}(x^2-x)\times 55$$

$$= 8x^2 - 6x + 1$$

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for error bound

$$|E_2(f;x)| = |f(x) - p(x)|$$

$$\le \frac{1}{6} M_3 \max_{0\le x\le 3} |x(x-1)(x-3)|$$

Let $w(x) = x(x-1)(x-3)$.

$\therefore w'(x) = (x-1)(x-3) + x(x-3) + x(x-1)$.

$\therefore w'(x) = 0$

$\Rightarrow (x-1)(x-3) + x(x-3) + x(x-1) = 0$

$\Rightarrow (x-3)(x-1+x) + x(x-1) = 0$

$\Rightarrow (x-3)(2x-1) + x(x-1) = 0$

$\Rightarrow 2x^2 - 6x - x + 3 + x^2 - x = 0$

$\Rightarrow 3x^2 - 8x + 3 = 0$

$\Rightarrow x = \frac{8 \pm \sqrt{64-36}}{6} = \frac{8 \pm \sqrt{28}}{6} = \ldots$

$x = 2.215250$

$ = 0.4514$

0.631130

$$\therefore \boxed{\begin{aligned} w''(x) = 6x - 8 &= 6\cdot(2.21525) - 8 > 0 \\ &= 6\cdot(0.4514) - 8 < 0 \end{aligned}}$$

$$\therefore \max_{0\le x\le 3} |x(x-1)(x-3)| = |2.215250(2.21525-1)(2.21525-3)|$$

$$= |-2.11261|$$

$$= 2.1126.$$

$$\therefore |E_2(f;x)| \le \frac{1}{6} \times 2.1126\, M_3 \qquad M_3 = \max_{0\le x\le 3} |f'''(x)|$$