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Numerical Differentiation

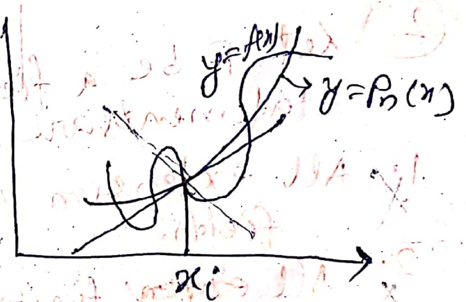
Numerical differentiation methods are obtained using one of the following three techniques.

(a) Methods based on interpolation.
(b) Methods based on finite difference operators.
(c) Methods based on undetermined coefficients.

we ~~now~~ discuss only (a) and (b).

Methods based on Interpolation:

Given the values of $f(x)$ at a set of points $x_0, x_1, \ldots x_n$, the general approach for deriving the numerical differentiation methods is to first obtain the interpolating polynomial $P_n(x)$ and then differentiate this polynomial r times $(n \geq r)$ to get $P_n^{(r)}(x)$. The value of $P_n^{(r)}(x_k)$ gives the approximate value of $f^{(r)}(x)$ at the nodal point x_k. It may be noted that though $P_n(x)$ and $f(x)$ have the same values at the nodal points, yet the derivatives may differ considerably at these points as in figure.



The situation may be worse at a non-nodal point. The approximation may further deteriorate as the order of derivative increases. The quantity

$$E^{(r)}(x) = f^{(r)}(x) - P_n^{(r)}(x)$$ is called the error of approximation in the r-th order derivative at any point x.

Non-uniform Nodal points:

If (x_i, f_i), $i = 0, 1, \ldots, n$ are $n+1$ distinct tabular points, then the Lagrange interpolating polynomial fitting this data is given by

$$P_n(x) = \sum_{k=0}^{n} l_k(x) f_k$$

where $l_k(x)$ is the Lagrange fundamental polynomial

$$l_k(x) = \frac{w(x)}{(x-x_k)\, w'(x_k)}$$

and $f_k = f(x_k)$, $w(x) = (x-x_0)(x-x_1)\cdots(x-x_n)$.

The error of approximation in (1) is given by

$$E_n(x) = f(x) - P_n(x)$$

$$= \frac{w(x)}{(n+1)!} f^{n+1}(\xi) \quad x_0 < \xi < x_n$$

for any point x. Differentiating (1) and (2) with respect to x, we obtain

$$P_n'(x) = \sum_{k=0}^{n} l_k'(x) f_k$$

and
$$E_n'(x) = \frac{w'(x)}{(n+1)!} f^{(n+1)}(\xi) + \frac{w(x)}{(n+1)!} \frac{d}{dx}\left(f^{(n+1)}(\xi)\right). \quad (3)$$

Since the function $\xi(x)$ in the second term on the right hand side of (3) is unknown, we cannot directly evaluate $E_n'(x)$. However, at a nodal point x_k, $w(x_k) = 0$ and we get

$$E_n'(x_k) = \frac{w'(x_k)}{(n+1)!} f^{(n+1)}(\xi), \quad x_0 < \xi < x_n$$

provided $\frac{d}{dx}\left(f^{(n+1)}(\xi)\right)$ remains bounded. For any r, $1 \le r \le n$, we obtain from (1)

$$f^{r}(x) \simeq P_n^{(r)}(x) = \sum_{k=0}^{n} l_k^{(r)} f_k$$

at any point x. The error term may be obtained by using the relation

$$\frac{1}{(n+1)} \frac{d^j}{dx^j}\left(f^{(n+1)}(\xi)\right) = \frac{j!}{(n+j+1)!} f^{(n+j+1)}(\eta_j) \quad j = 1, 2 \ldots$$

where $\min(x_0, x_1, \ldots x_n, x) < \eta_j < \max(x_0, x_1, \ldots x_n, x)$.