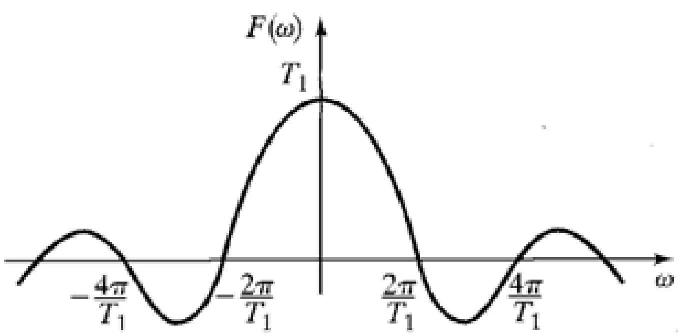
Fourier Transform

Basic Idea

- We covered the Fourier Transform which to represent periodic signals
- We assumed periodic continuous signals

$$f_p(t) = \sum_k C_k e^{jk\omega_0 t}, \quad C_k = \frac{1}{T_0}\int_{T_0} f_p(t)e^{-jk\omega_0 t}dt$$

- We used Fourier Series to represent periodic continuous time signals in terms of their harmonic frequency components (Ck).
- We want to extend this discussion to find the frequency spectra of a given signal



Basic Idea

- The Fourier Transform is a method for representing signals and systems in the frequency domain
- We start by assuming the period of the signal is T=→ INF
- All physically realizable signals have Fourier Transform

- For aperiodic signals Fourier Transform pairs is described as

$$\text{ANALYSIS } F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$\text{SYNTHESIS } f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

Fourier Transforms of f(t)

Inverse Fourier Transforms of F(ω)

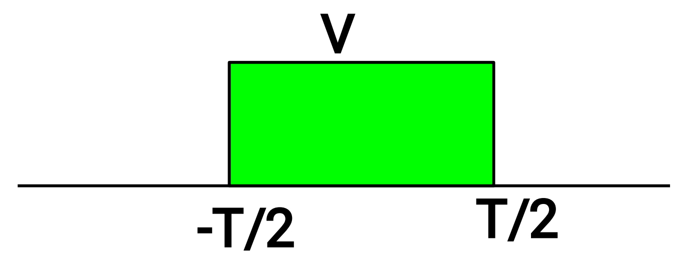
Remember:

$$f_p(t) = \sum_k C_k e^{jk\omega_0 t}, \quad C_k = \frac{1}{T_0} \int_{T_0} f_p(t) e^{-jk\omega_0 t} dt$$

notes

Example – Rectangular Signal

- Compute the Fourier Transform of an aperiodic rectangular pulse of T seconds evenly distributed about t=0.



$$f(t) = Vu(t + T/2) - Vu(t - T/2)$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

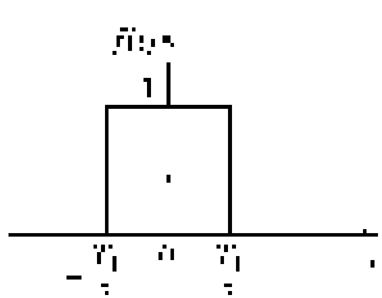
$$= TV \sin c(\omega T/2)$$

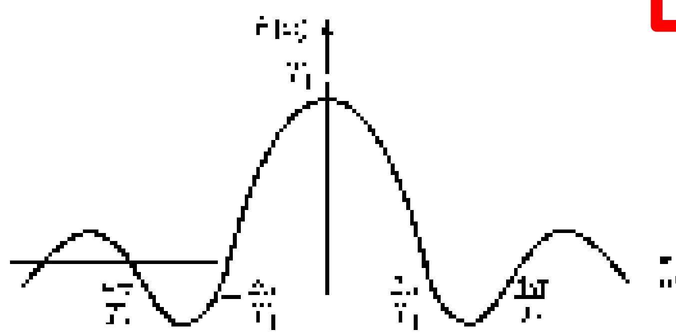
notes

thus

$$V rect(t/T) \overset{\mathfrak{I}}{\longleftrightarrow} TV \sin c(\omega T/2)$$

- Remember this the same rectangular signal as we worked before but with T0→ infinity!





All physically realizable signals have Fourier Transforms

Fourier Transform of Unit Impulse Function

$$f(t) = A\,\delta(t - t_0)$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} A\,\delta(t - t_0)\, e^{-j\omega t}\, dt = Ae^{-j\omega t_0}$$

$$if\ :\ f(t) = A\,\delta(t-0) \rightarrow F(\omega) = A$$

thus

$$\delta(t) \overset{\Im}{\longleftrightarrow} A$$

Example:

$$F(\omega) = \delta(\omega - \omega_0)$$

$$\Im^{-1}\{F(\omega)\} = f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_0)\, e^{j\omega t}\, d\omega$$

$$= \frac{e^{j\omega_0 t}}{2\pi}$$

thus

$$e^{j\omega_0 t} \overset{\Im}{\longleftrightarrow} 2\pi\delta(\omega - \omega_0)$$

Plot magnitude and phase of f(t)

Fourier Series Properties

Operation	Time Function	Fourier Transform
Linearity	$af_1(t) + bf_2(t)$	$aF_1(\omega) + bF_2(\omega)$
Time shift	$f(t - t_0)$	$F(\omega)e^{-j\omega t_0}$
Time scaling	$f(at)$	$\frac{1}{\lvert a\rvert}F\left(\frac{\omega}{a}\right)$
Time transformation	$f(at - t_0)$	$\frac{1}{\lvert a\rvert}F\left(\frac{\omega}{a}\right)e^{-j\omega t_0/a}$
Duality	$F(t)$	$2\pi f(-\omega)$
Frequency shift	$f(t)e^{j\omega_0 t}$	$F(\omega-\omega_0)$
Convolution	$f_1(t)*f_2(t)$	$F_1(\omega)F_2(\omega)$
	$f_1(t)f_2(t)$	$\frac{1}{2\pi}F_1(\omega)*F_2(\omega)$
Differentiation	$\frac{d^n[f(t)]}{dt^n}$	$(j\omega)^n F(\omega)$
	$(-jt)^n f(t)$	$\frac{d^n[F(\omega)]}{d\omega^n}$
Integration	$\int_{-\infty}^{t} f(\tau)d\tau$	$\frac{1}{j\omega}F(\omega) + \pi F(0)\delta(\omega)$

Make sure how to use these properties!

Fourier Series Properties - Linearity

$$a f_1(t) + b f_2(t) \qquad\qquad a F_1(\omega) + b F_2(\omega)$$

$$f(t) = B\cos(\omega_0 t)$$

Find F(w)

Fourier Series Properties - Linearity

$$a f_1(t) + b f_2(t) \qquad\qquad a F_1(\omega) + b F_2(\omega)$$

$$f(t) = B\cos(\omega_0 t)$$

$$= B/2(e^{j\omega_0 t} + e^{-j\omega_0 t})$$

$$\Im\{f(t)\} = F(\omega) = \frac{B}{2}\Im\{e^{j\omega_0 t}\} + \frac{B}{2}\Im\{e^{-j\omega_0 t}\}$$

Due to linearity

$$\text{Re } member: e^{j\omega_0 t} \overset{\Im}{\longleftrightarrow} 2\pi\delta(\omega-\omega_0)$$

$$\Rightarrow \frac{B}{2}2\pi\delta(\omega-\omega_0) + \frac{B}{2}2\pi\delta(\omega+\omega_0)$$

Fourier Series Properties - Time Scaling

$g(t) = rect\,(2t/T_1)$

Re *member* :

$f(t) = Vrect\;(t/T_1) \xleftrightarrow{\Im} F(\omega) = T_1 V \sin\; c(T_1\omega/2)$

Thus

$g(t) \xleftrightarrow{\Im} G(w)$

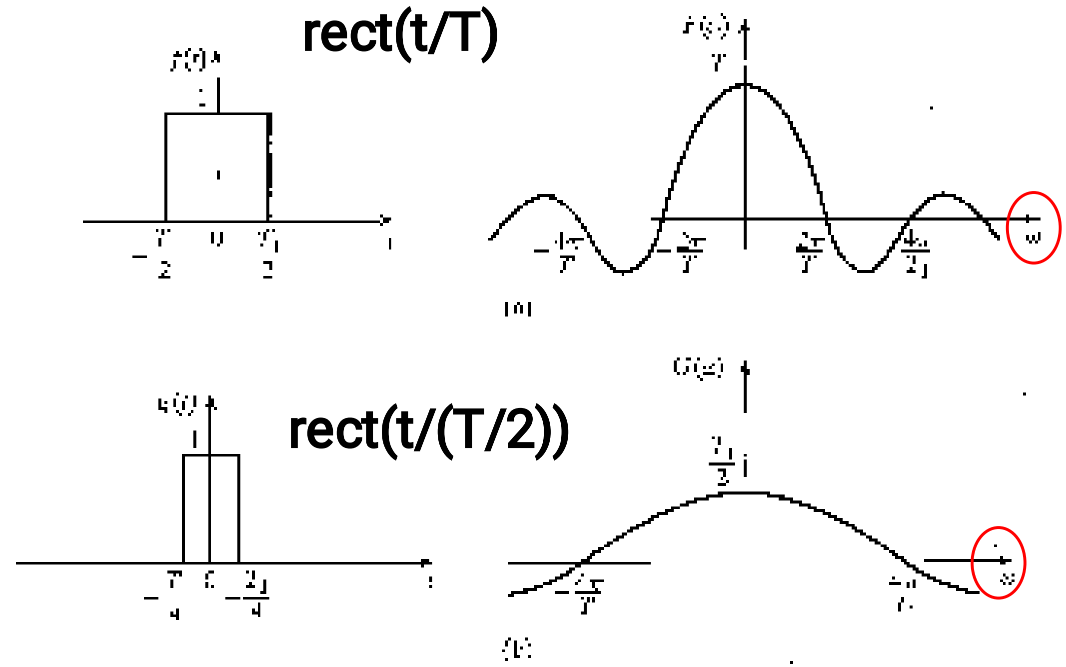
$g(t) = \frac{1}{V}\; f(at)\Big|_{a=2} \xleftrightarrow{\Im} G(w) = 1/|a|\, F(\omega/a)$

Then

$G(\omega) = \frac{1}{V}(\frac{1}{2})(T_1 V)\sin\; c(T_1\omega/4)$

$= (\frac{T_1}{2})\sin\; c(T_1\omega/4)$

Due to Time Scaling Property



Remember:
sinc(0)=1;
sinc(2pi)=0=sinc(pi)

Fourier Series Properties - Duality or Symmetry

$$x(t) \leftrightarrow X(\omega)$$
$$X(t) \leftrightarrow 2\pi\, x(-\omega)$$

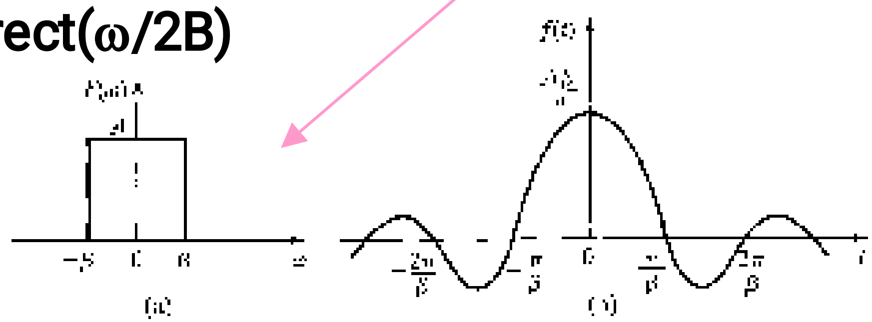
$$if\ :\ f(t) \overset{\Im}{\longleftrightarrow} F(\omega)$$

$$F(t) \overset{\Im}{\longleftrightarrow} 2\pi f(-\omega)$$

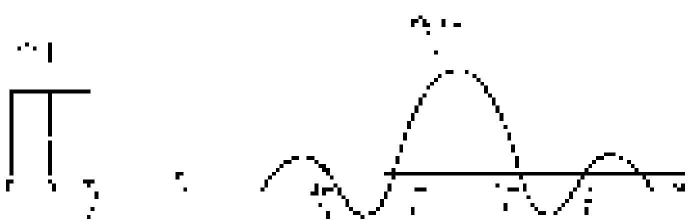
$$\Rightarrow \Im\{F(t)\} = 2\pi f(-\omega)$$

$$where\ :\ f(-\omega) = f(t)\Big|_{t=-\omega}$$

Arect(ω/2B)



Remember we had:



Example:
Find the time-domain waveform for

$$F(\omega) = Au(\omega + B) - Au(\omega - B)$$

$$Re\ member\ :$$

$$Vrect\ (t/T) \overset{\Im}{\longleftrightarrow} TV\ \sin\ c(\omega T/2)$$

$$We\_have\ :$$

$$F(\omega) = Au(\omega + B) - Au(\omega - B) = Arect\ (\omega/2B)$$

$$U\ \sin\ g\_Duality\ \_find\ \_F(t)$$

$$2\pi f(-\omega) = 2\pi Arect\ (-\omega/T)$$

$$= 2\pi Arect\ (-\omega/2B)$$

$$Thus$$

$$F(t) = 2BA\ \sin\ c(Bt)$$

$$BA\ \sin\ c(Bt) \overset{\Im}{\longleftrightarrow} \pi Arect\ (-\omega/2B)$$

Refer to FTP Table

FTP: Fourier Transfer Pair

Fourier Series Properties - Duality or Symmetry

Example: find the frequency response
Of y(t)

$$y(t) = \frac{1}{a + jt}$$

Fourier Series Properties - Duality or Symmetry

Example: find the frequency response
Of y(t)

$$y(t) = \frac{1}{a + jt}$$

$$if \ : \ f(t) \overset{\Im}{\longleftrightarrow} F(\omega)$$

$$F(t) \overset{\Im}{\longleftrightarrow} 2\pi f(-\omega)$$

$$\Rightarrow \Im\{F(t)\} = 2\pi f(-\omega)$$

$$where \ : \ f(-\omega) = f(t)\Big|_{t=-\omega}$$

We know
Using Fourier Transform Pairs

$$x(t) = e^{-at}u(t), \ a > 0$$

$$X(\omega) = \frac{1}{a + j\omega}$$

Using duality

$$e^{-at}u(t) \leftrightarrow \frac{1}{a + j\omega}$$

$$\frac{1}{a + jt} \leftrightarrow 2\pi e^{a\omega}u(-\omega)$$

Fourier Series Properties - Convolution

$$x(t) * h(t) \leftrightarrow X(\omega)H(\omega)$$

Proof

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

$$Y(\omega) = \int_{-\infty,dt}^{\infty} y(t)e^{-j\omega t}dt = \int_{-\infty,dt}^{\infty} \int_{-\infty,d\tau}^{\infty} x(\tau)h(t-\tau)d\tau \; e^{-j\omega t}dt$$

$$Y(\omega) = \int_{-\infty,d\tau}^{\infty} x(\tau) \int_{-\infty,dt}^{\infty} h(t-\tau)e^{-j\omega t}dt\,d\tau \quad \textbf{let} \;\; u = t - \tau$$

$$= \int_{-\infty,d\tau}^{\infty} x(\tau) \int_{-\infty,du}^{\infty} h(u)e^{-j\omega(u+\tau)}du\,d\tau$$

$$= \int_{-\infty,d\tau}^{\infty} x(\tau)e^{-j\omega\tau}d\tau \int_{-\infty,du}^{\infty} h(u)e^{-j\omega u}du$$

$$= X(\omega)H(\omega)$$

$$y(t) = x(t) * h(t) \leftrightarrow Y(\omega) = X(\omega)H(\omega)$$

$$x_1(t)x_2(t) \leftrightarrow \frac{1}{2\pi} X_1(\omega) * X_2(\omega)$$

Proof

$$x_1(t)x_2(t) \leftrightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(u)X_2(\omega - u)du$$

$$F[x_1(t)x_2(t)] = \int_{-\infty}^{\infty} x_1(t)x_2(t)e^{-j\omega t}dt$$

$$F[x_1(t)x_2(t)] = \int_{-\infty,dt}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty,da}^{\infty} X_1(a)e^{-jat}da \right] x_2(t)e^{-j\omega t}dt$$

$$= \frac{1}{2\pi} \int_{-\infty,da}^{\infty} X_1(a) \int_{-\infty,dt}^{\infty} x_2(t)e^{-j(\omega - a)t}dt\,da$$

$$= \frac{1}{2\pi} \int_{-\infty,da}^{\infty} X_1(a)X_2(\omega - a)da$$

$$= \frac{1}{2\pi} X_1(\omega) * X_2(\omega)$$

Fourier Series Properties - Convolution

Example:
Find the Fourier Transform of x(t)=sinc²(t)

$$x(t) \leftrightarrow X(\omega)$$
$$X(t) \leftrightarrow 2\pi\, x(-\omega)$$

$$\frac{1}{\pi}\text{sinc}(t) \leftrightarrow \text{rect}(\frac{\omega}{2})$$

$$\text{sinc}(t) \leftrightarrow \pi\,\text{rect}(\frac{\omega}{2})$$

$$x_1(t)x_2(t) \leftrightarrow \frac{1}{2\pi} X_1(\omega) * X_2(\omega)$$

$$\text{sinc}^2(t) \leftrightarrow \frac{1}{2\pi}(\pi\,\text{rect}(\frac{\omega}{2}) * \pi\,\text{rect}(\frac{\omega}{2}))$$

Re *member* :

$$BA \sin c(Bt) \overset{\Im}{\longleftrightarrow} \pi A\,rect\,(-\omega/2B)$$

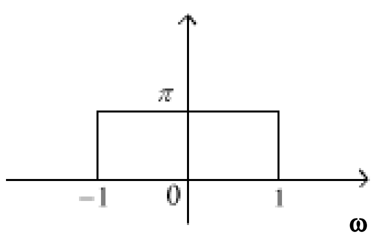
$$\sin c(t) \overset{\Im}{\longleftrightarrow} \pi\,rect\,(-\omega/2)$$

$$\sin c(t) \overset{\Im}{\longleftrightarrow} \pi\,rect\,(\omega/2)$$

In this case we have B=1, A=1

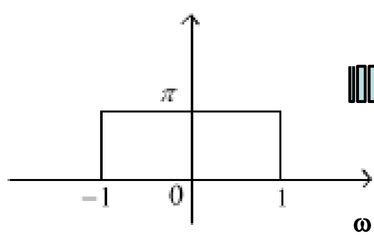
$$X(0) = \frac{1}{2\pi}\int_{-1}^{1} \pi^2 d\omega$$

$$= \frac{1}{2\pi}(2\pi^2) = \pi$$

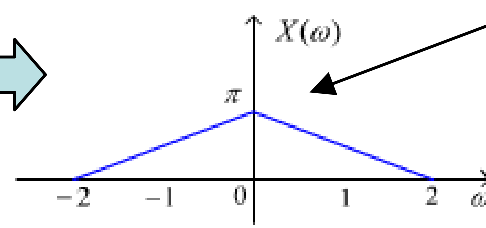


$X_1(\omega)$

*



$X_2(\omega)$



Refer to Schaum's Prob. 2.6

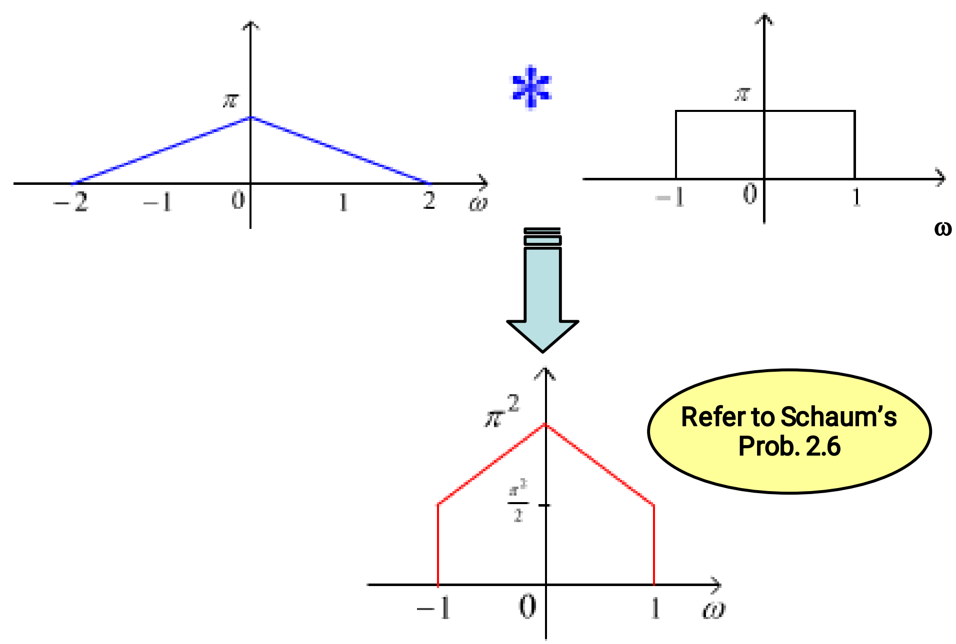
Fourier Series Properties - Convolution

Example:
Find the Fourier Transform of x(t)= sinc^2(t) sinc(t)

$$\text{sinc}(t) \leftrightarrow \text{rect function}$$

$$\text{sinc}^2(t) \leftrightarrow \text{triangle function}$$

We need to find the convolution of a rect and a triangle function:



Fourier Series Properties - Frequency Shifting

$$x(t)\,e^{j\omega_0 t} \overset{\Im}{\longleftrightarrow} X(\omega-\omega_0)$$

Example:
Find the Fourier Transform of g3(t) if g1(t)=2cos(200πt), g2(t)=2cos(1000πt); g3(t)=g1(t).g2(t) ; that is [G3(w)]

$$g_3(t) = 5\cos(200\pi t)\,e^{j1000\pi t} + 5\cos(200\pi t)\,e^{-j1000\pi t}$$

$$\Rightarrow G_3(\omega) = 5\pi\delta(\omega+200\pi-1000\pi) + 5\pi\delta(\omega-200\pi-1000\pi)$$

$$+5\pi\delta(\omega+200\pi+1000\pi) + 5\pi\delta(\omega-200\pi+1000\pi)$$

$$= 5\pi\delta(\omega-800\pi) + 5\pi\delta(\omega-1200\pi)$$

$$+5\pi\delta(\omega+1200\pi) + 5\pi\delta(\omega+800\pi)$$

Remember: cosa . cosb=1/2[cos(a+b)+cos(a-b)]

Fourier Series Properties - Time Differentiation

$$f(t) \overset{\mathfrak{I}}{\longleftrightarrow} F(\omega)$$

$$g(t) = df(t)/dt \overset{\mathfrak{I}}{\longleftrightarrow} j\omega F(\omega)$$

Also

$$g(t) = \int_{-\infty}^{t} f(\tau)\, d\tau \overset{\mathfrak{I}}{\longleftrightarrow} \frac{1}{j\omega} F(\omega) + \pi F(0)\delta(\omega) = G(\omega)$$

Note

$$F(0) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} f(t)\, dt$$

Example:

$$f(t) = \operatorname{sgn}(t)$$

$$g(t) = u(t) \longrightarrow dg(t)/dt = \delta(t)$$

$$df(t)/dt = 2\delta(t)$$

$$\delta(t) \overset{\mathfrak{I}}{\longleftrightarrow} 1$$

thus

$$\operatorname{sgn}(t) = df(t)/dt \overset{\mathfrak{I}}{\longleftrightarrow} 1$$