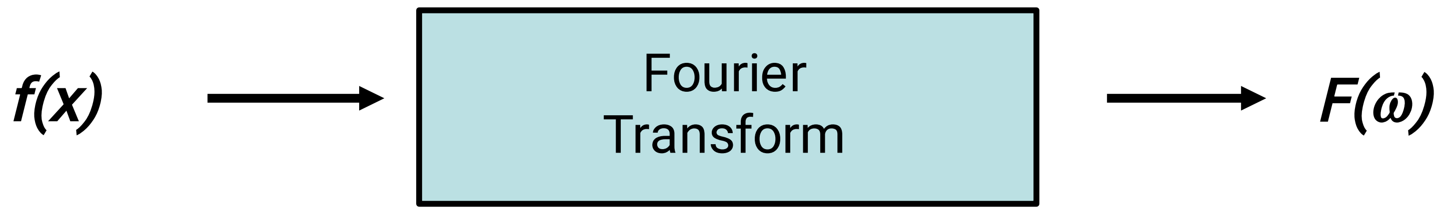
Fourier Transform

- We want to understand the frequency ω of our signal. So, let's reparametrize the signal by ω instead of x :



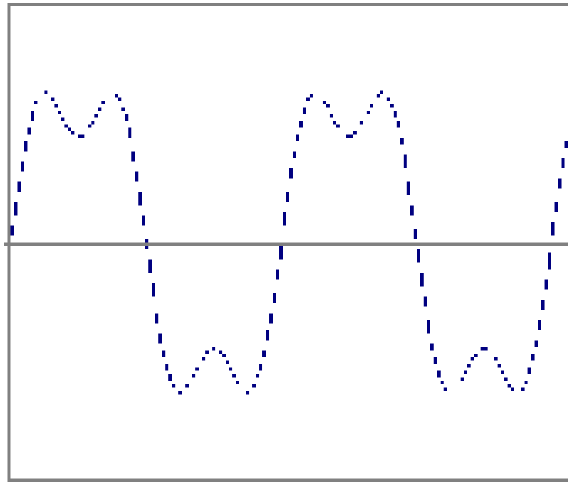
- For every ω from 0 to inf, ***F(ω)*** holds the amplitude A and phase ϕ of the corresponding sine $A\sin(\omega x + \phi)$
  - How can F hold both? Complex number trick!

$$F(\omega) = R(\omega) + iI(\omega)$$

$$A = \pm\sqrt{R(\omega)^2 + I(\omega)^2} \qquad \phi = \tan^{-1}\frac{I(\omega)}{R(\omega)}$$

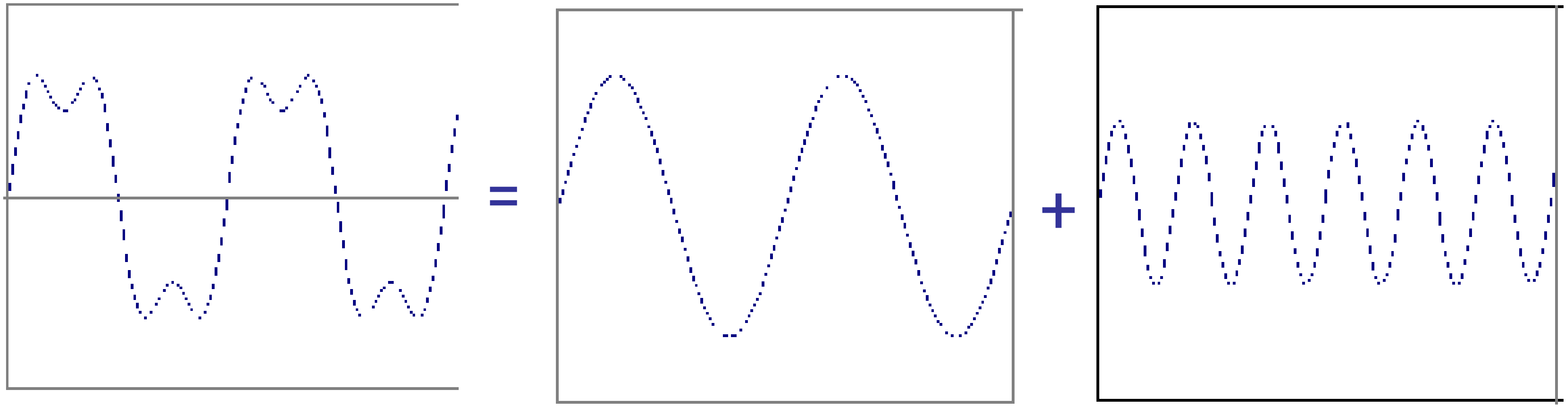
Time and Frequency

- example : $g(t) = \sin(2pi\ f\ t) + (1/3)\sin(2pi\ (3f)\ t)$



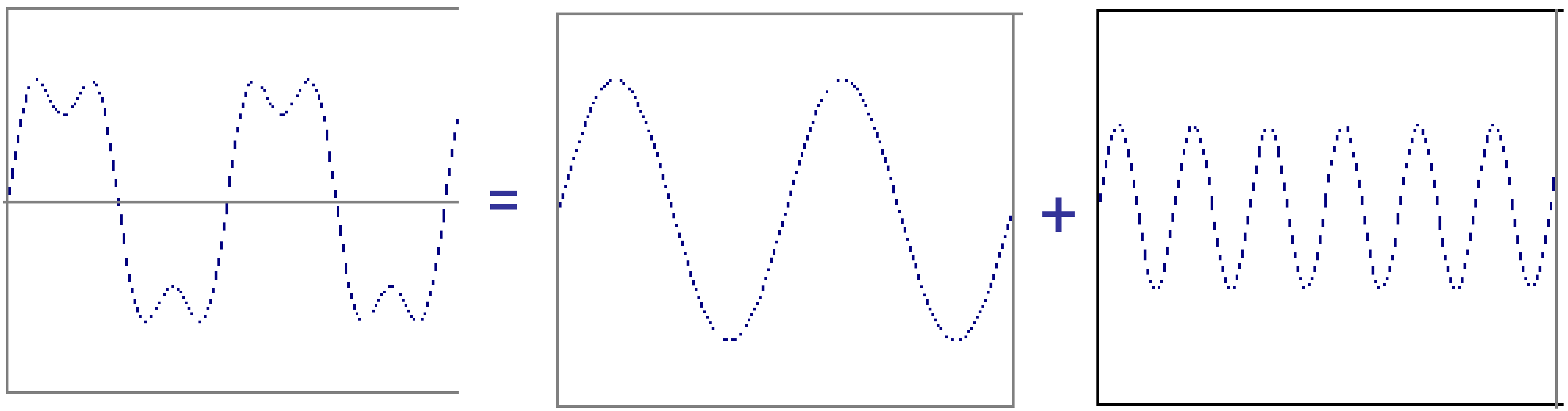
Time and Frequency

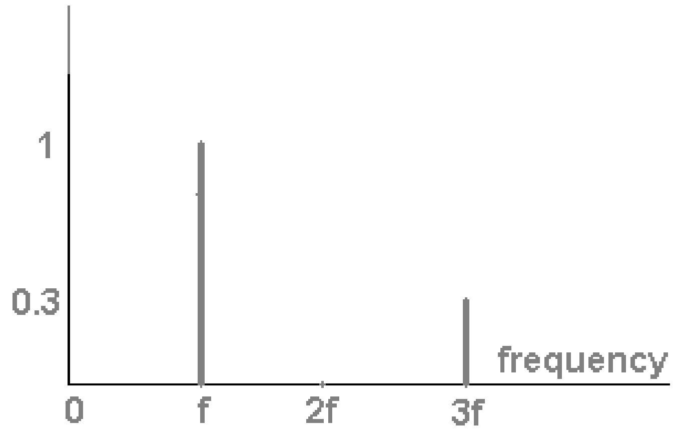
- example : $g(t) = \sin(2\pi f\, t) + (1/3)\sin(2\pi\,(3f)\,t)$



Frequency Spectra

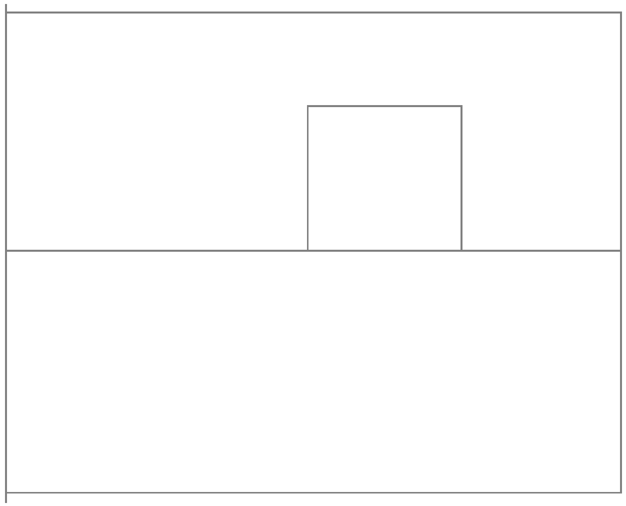
- example : $g(t) = \sin(2\pi f t) + (1/3)\sin(2\pi (3f) t)$



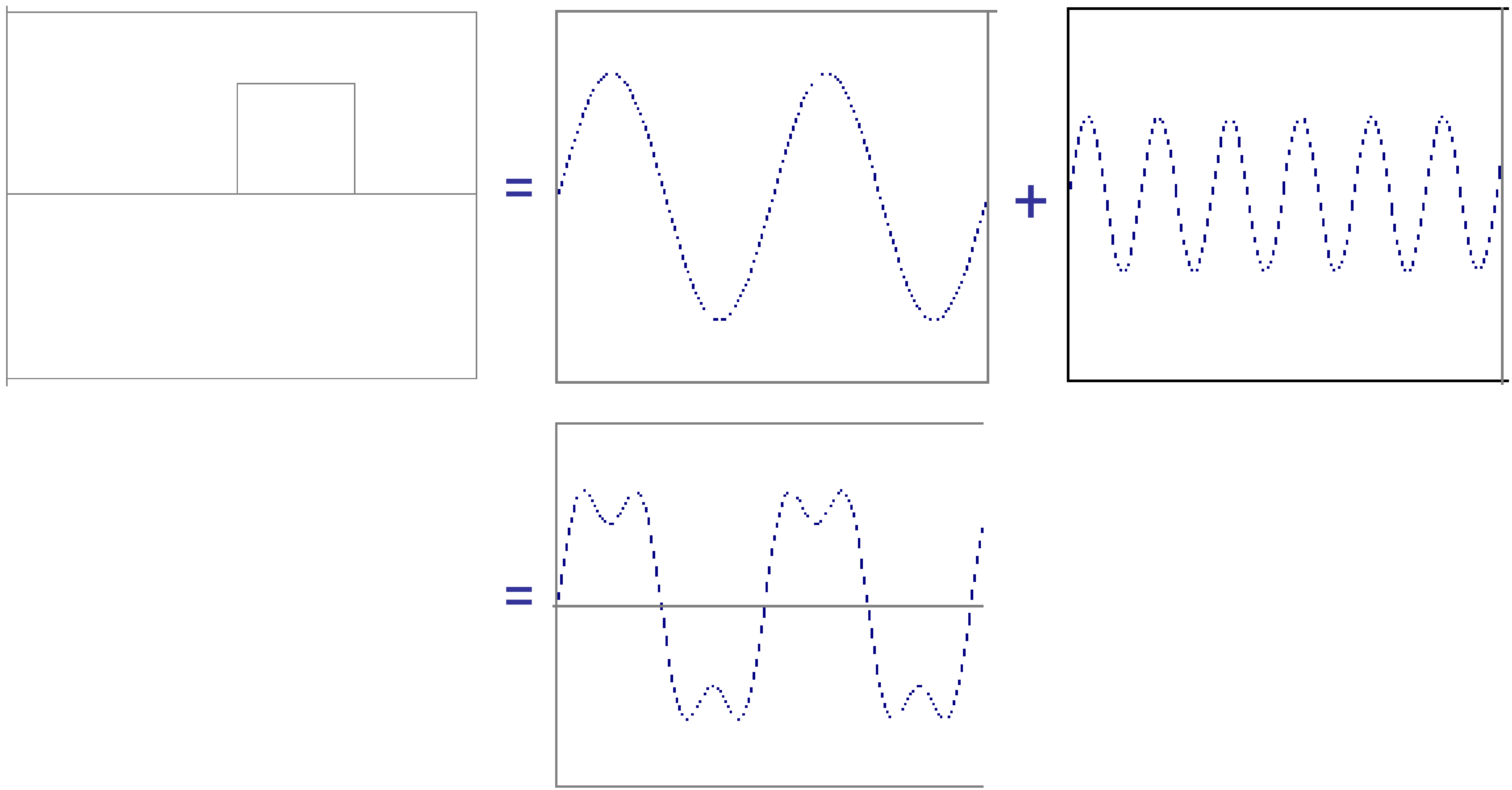


Frequency Spectra

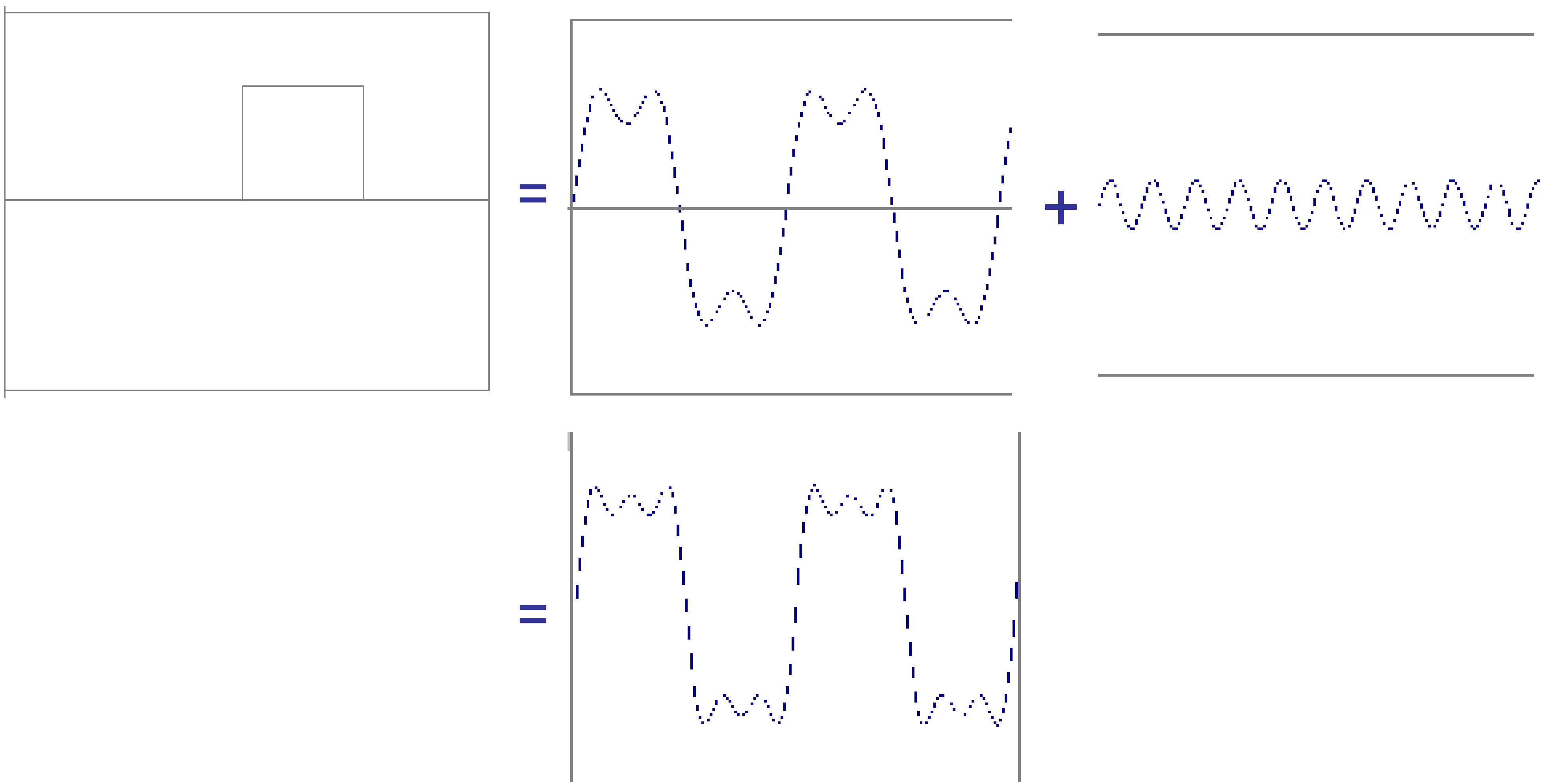
- Usually, frequency is more interesting than the phase



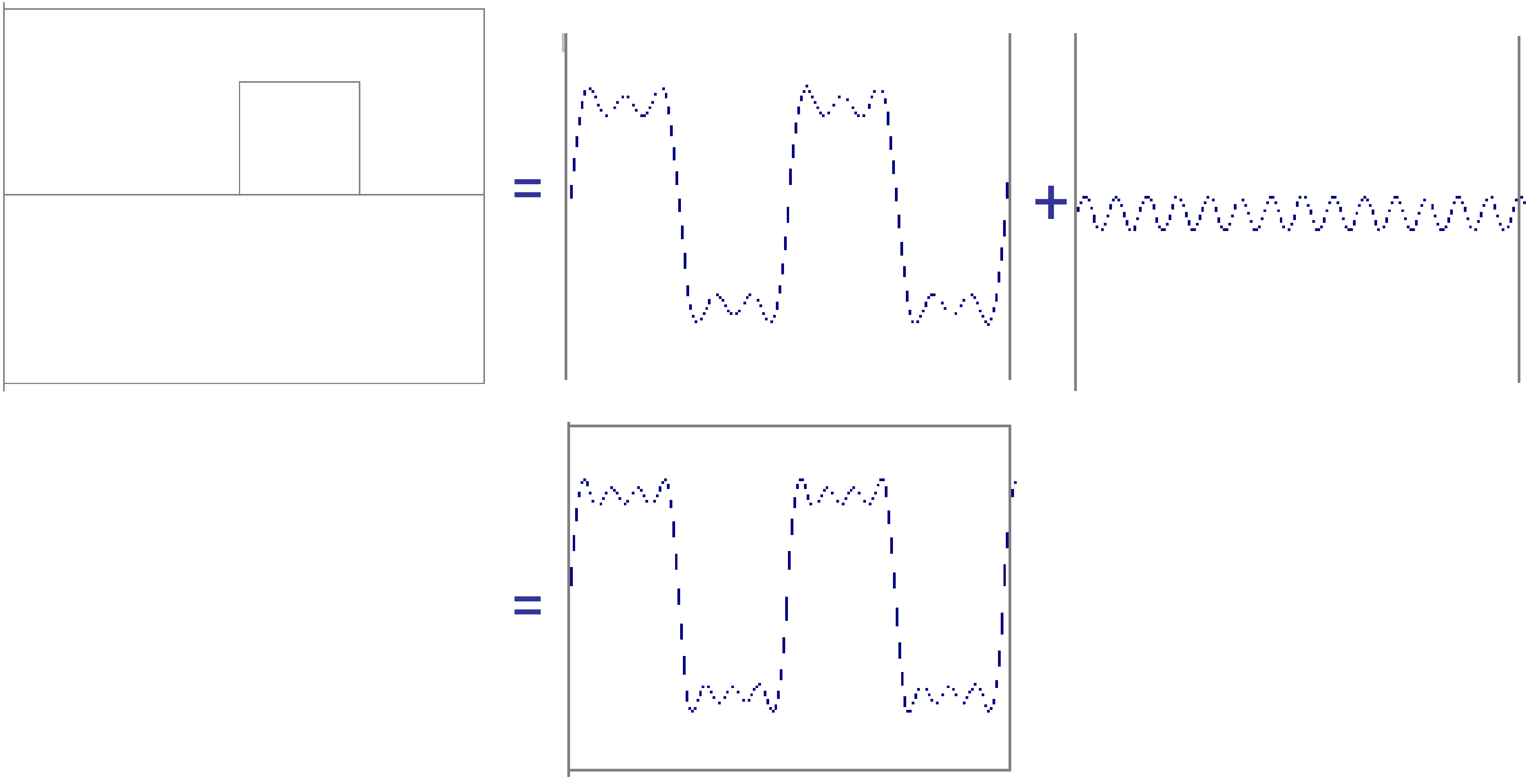
Frequency Spectra



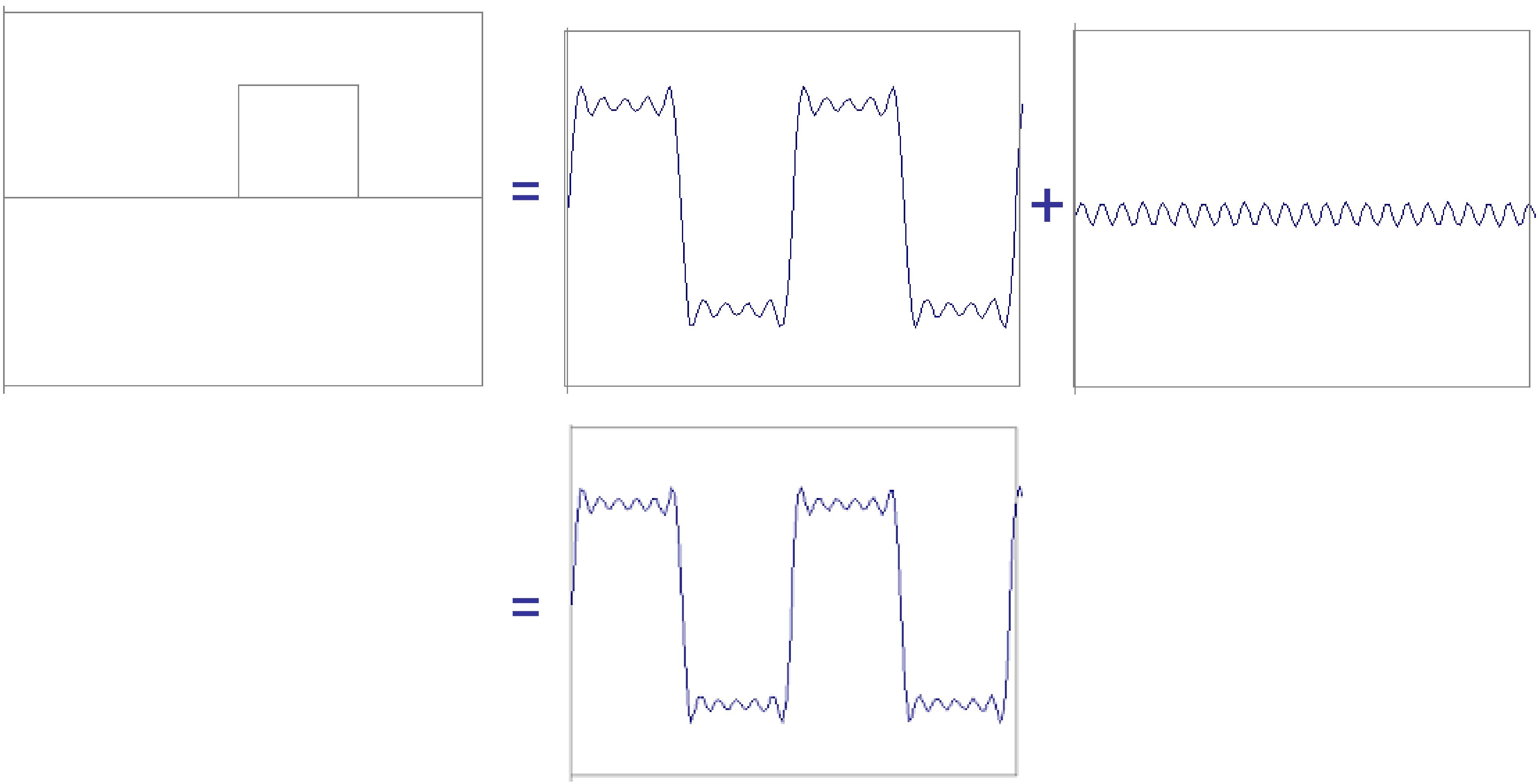
Frequency Spectra



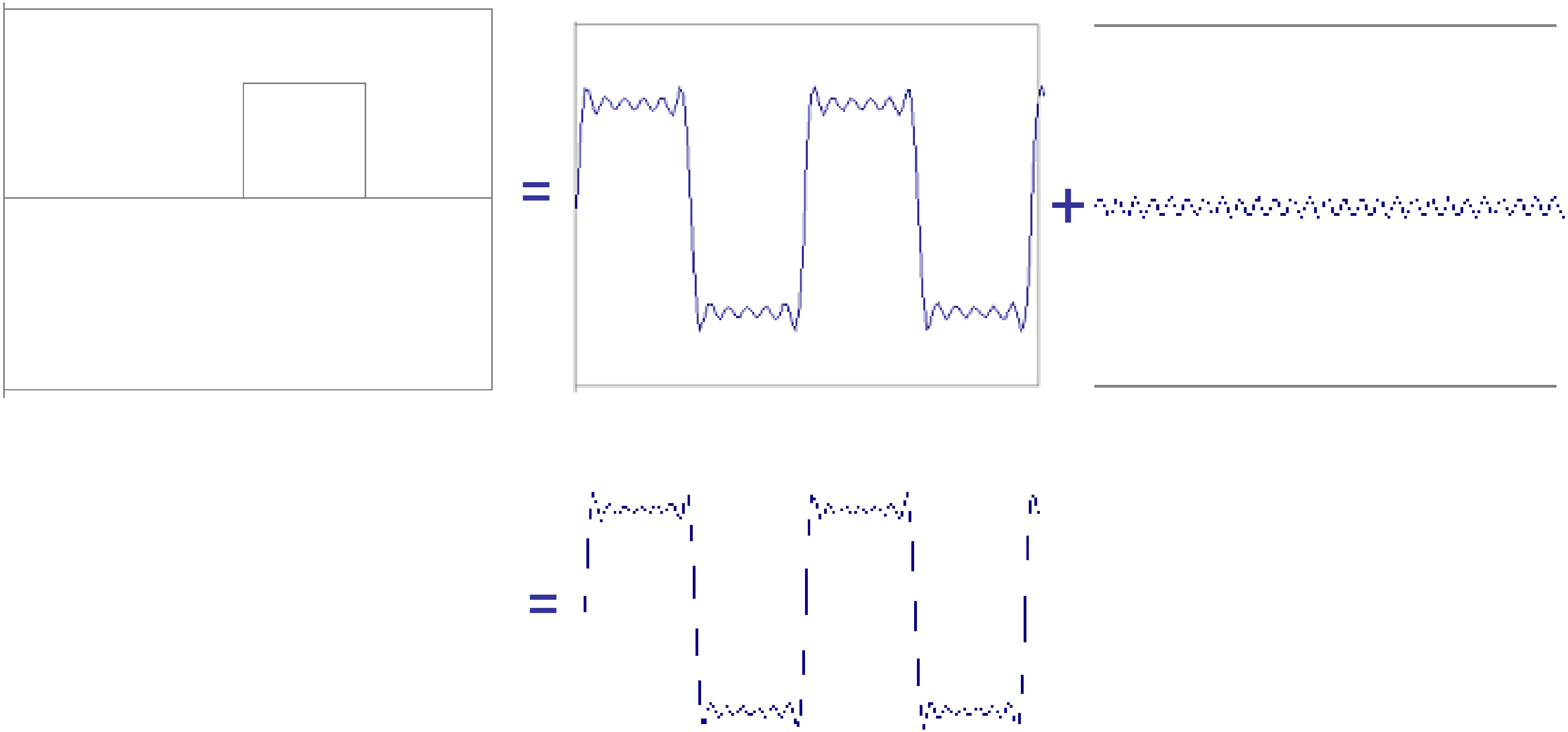
Frequency Spectra



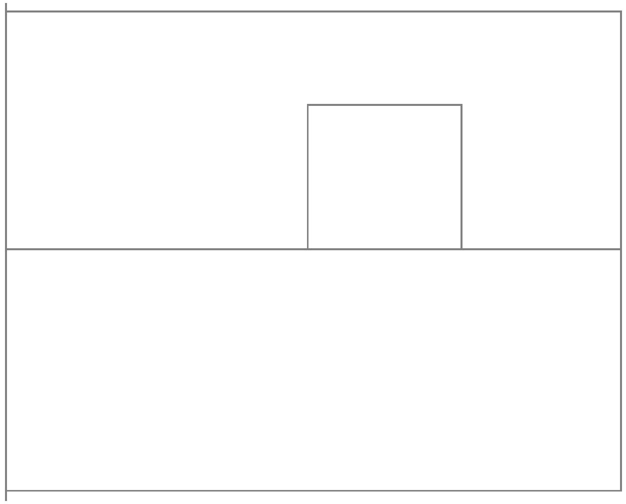
Frequency Spectra



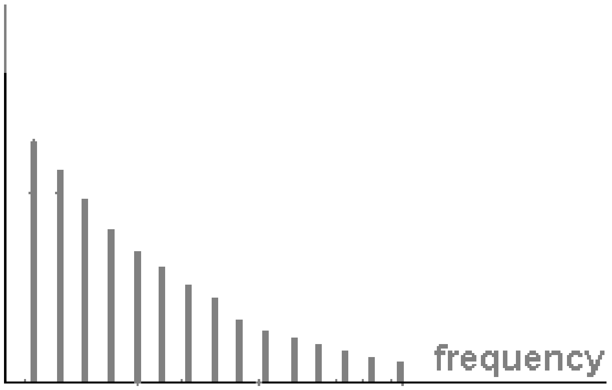
Frequency Spectra



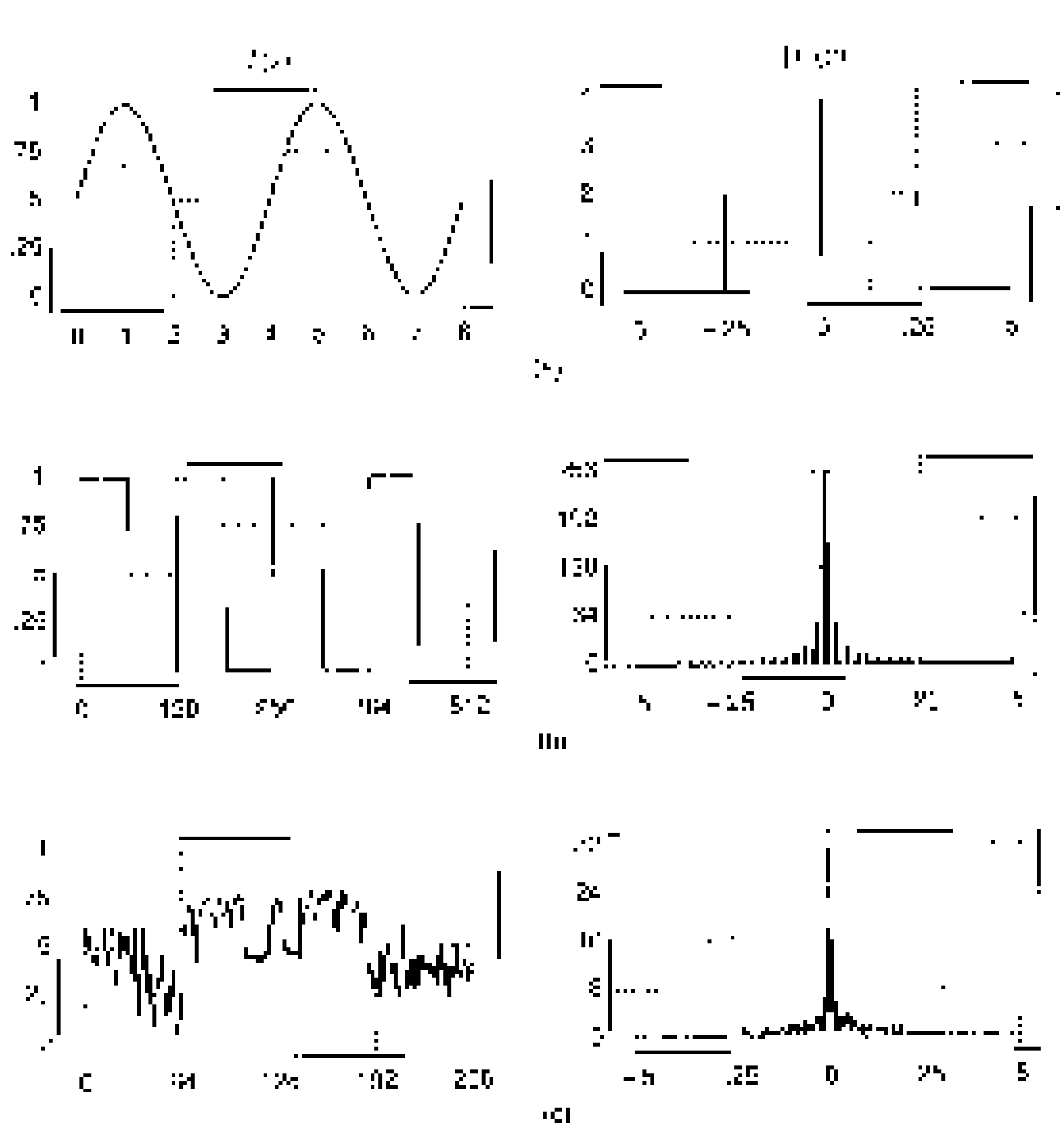
Frequency Spectra



$$= A\sum_{k=1}^{\infty}\frac{1}{k}\sin(2\pi kt)$$

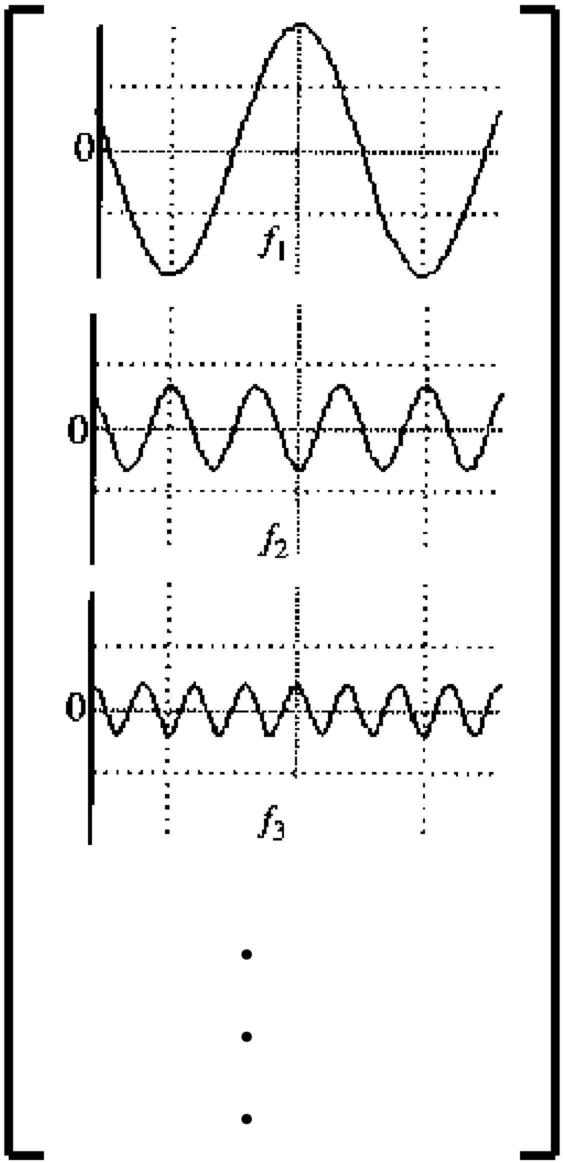


Frequency Spectra



FT: Just a change of basis

$$M * f(x) = F(\omega)$$



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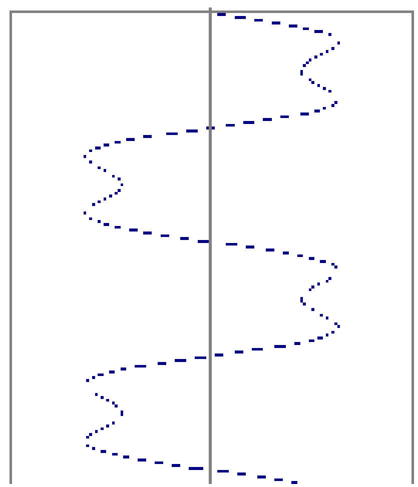

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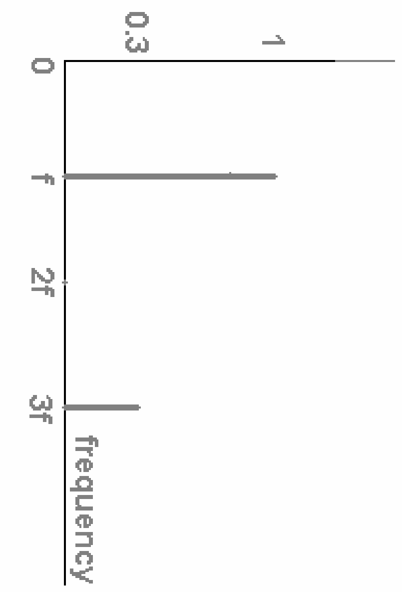
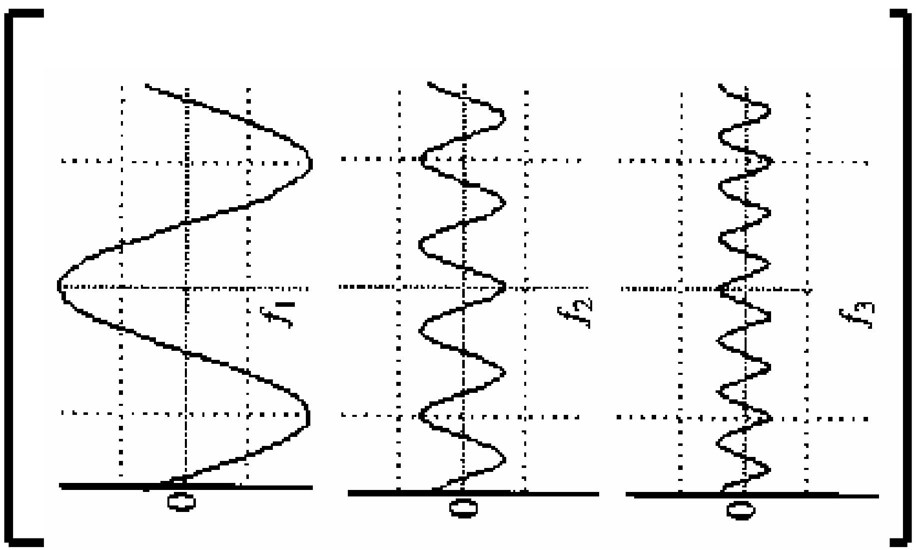


IFT: Just a change of basis

$$M^{-1} * F(\omega) = f(x)$$



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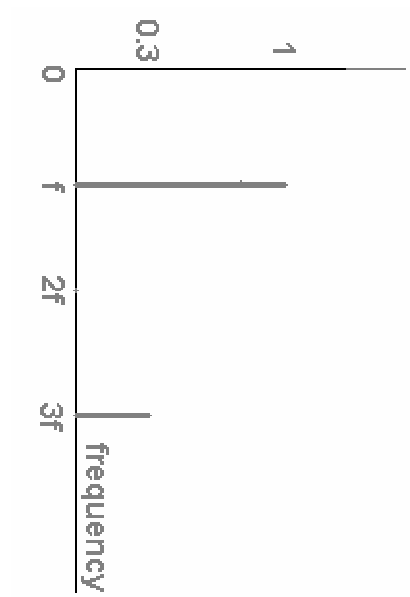

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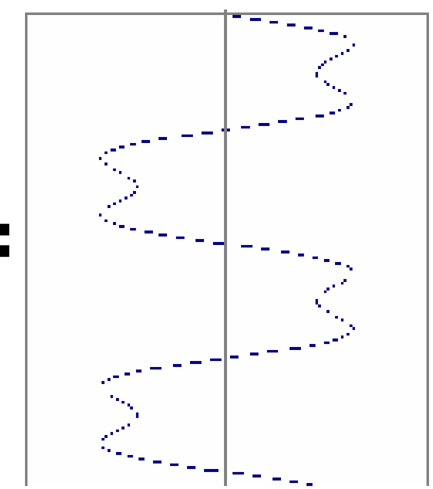


Fourier Transform – more formally

Represent the signal as an infinite weighted sum of an infinite number of sinusoids

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-i 2\pi ux} dx$$

Note: $e^{ik} = \cos k + i \sin k \qquad i = \sqrt{-1}$

Arbitrary function ⟶ Single Analytic Expression

Spatial Domain (x) ⟶ Frequency Domain (u)

(Frequency Spectrum $F(u)$)

Inverse Fourier Transform (IFT)

$$f(x) = \int_{-\infty}^{\infty} F(u) e^{i 2\pi ux} dx$$

Fourier Transform

- Also, defined as:

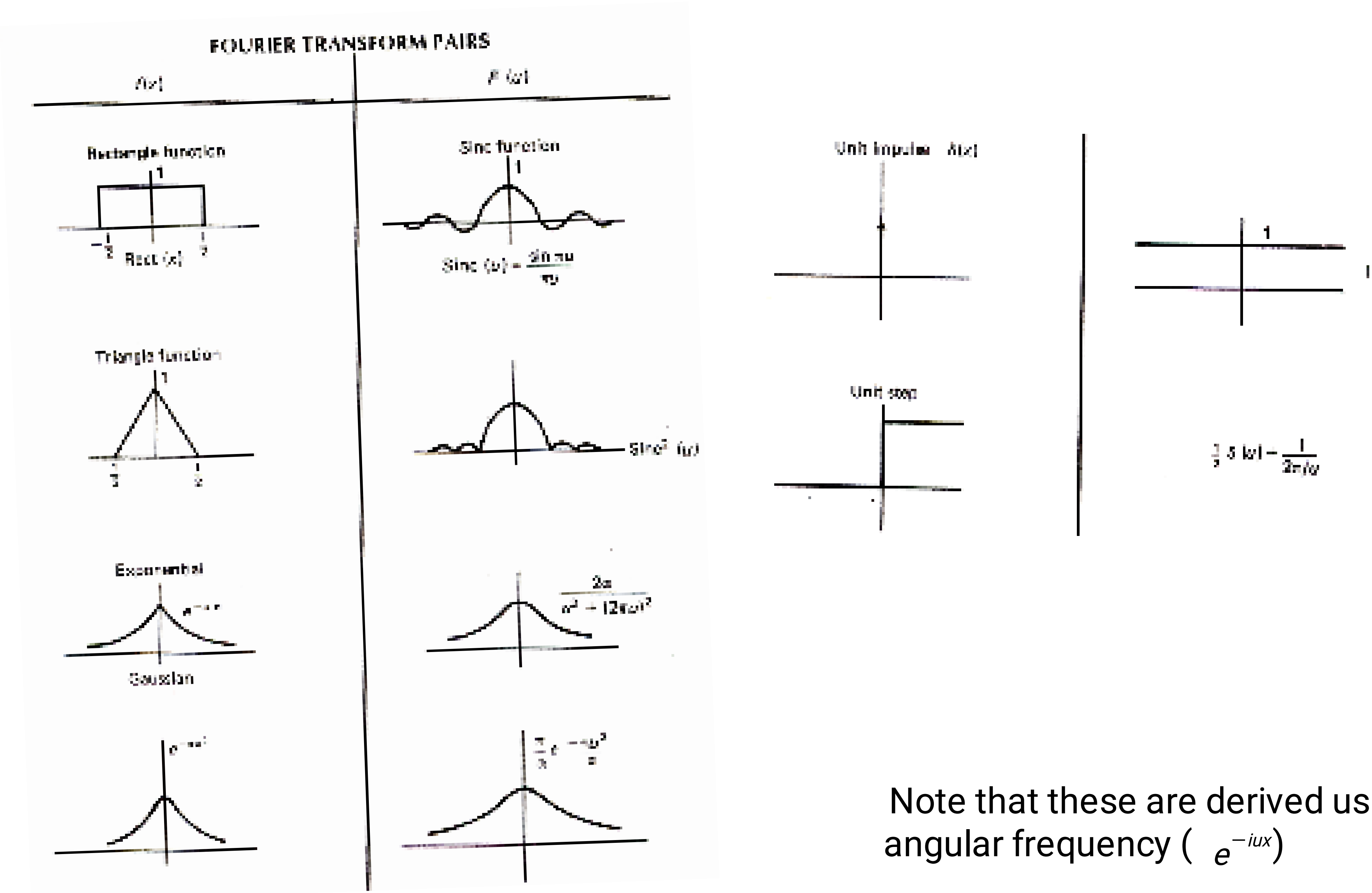
$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-iux} \, dx$$

Note: $e^{ik} = \cos k + i \sin k \qquad i = \sqrt{-1}$

- Inverse Fourier Transform (IFT)

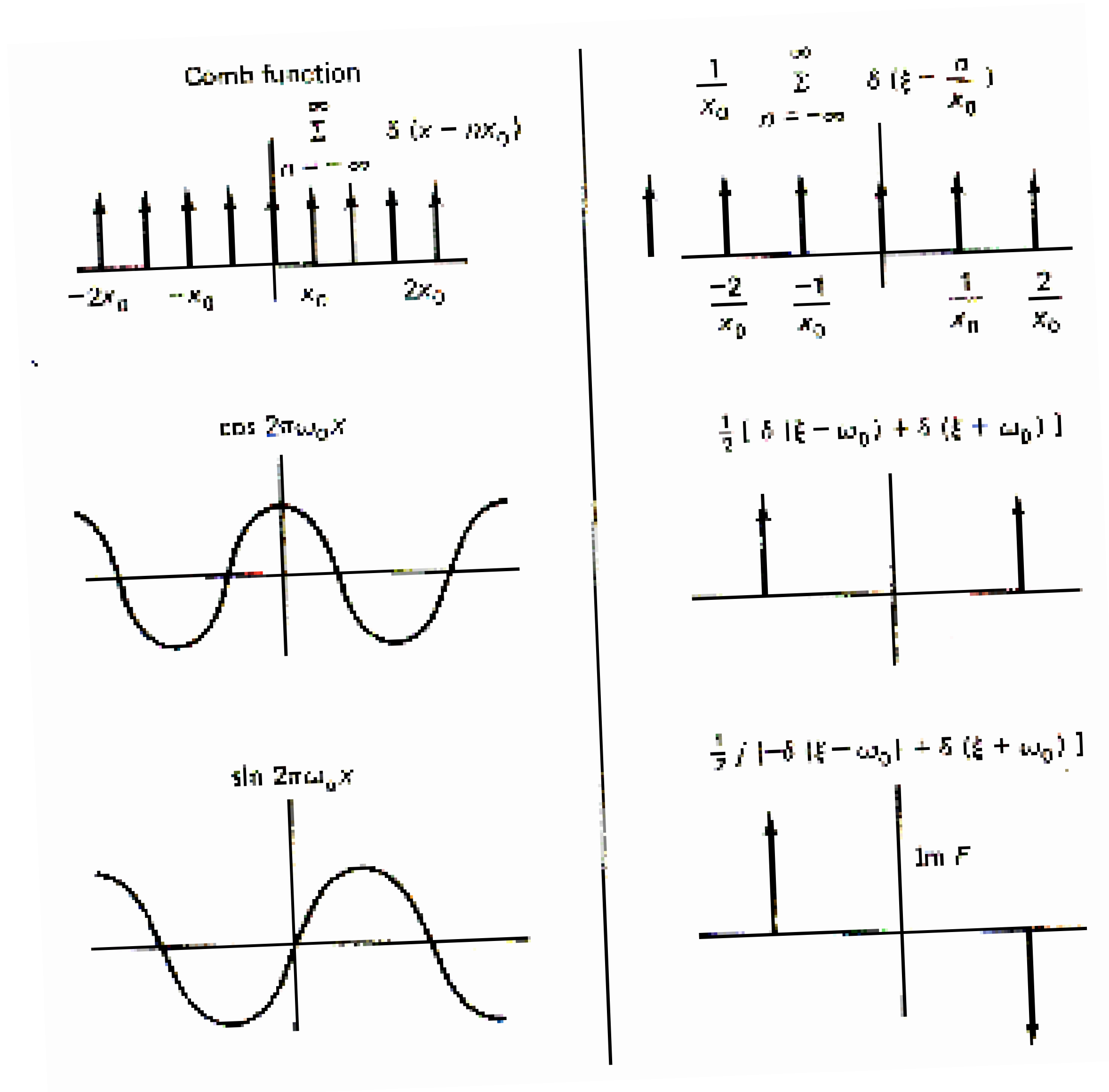
$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(u) e^{iux} \, dx$$

Fourier Transform Pairs (I)



Note that these are derived using angular frequency (e^{-iux})

Fourier Transform Pairs (I)



Note that these are derived using angular frequency (e^{-iux})

Fourier Transform and Convolution

Let $g = f * h$

Then $G(u) = \int_{-\infty}^{\infty} g(x) e^{-i2\pi ux} dx$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau) h(x-\tau) e^{-i2\pi ux} d\tau dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [f(\tau) e^{-i2\pi u\tau} d\tau][h(x-\tau) e^{-i2\pi u(x-\tau)} dx]$$

$$= \int_{-\infty}^{\infty} [f(\tau) e^{-i2\pi u\tau} d\tau] \int_{-\infty}^{\infty} [h(x') e^{-i2\pi ux'} dx']$$

$$= F(u) H(u)$$

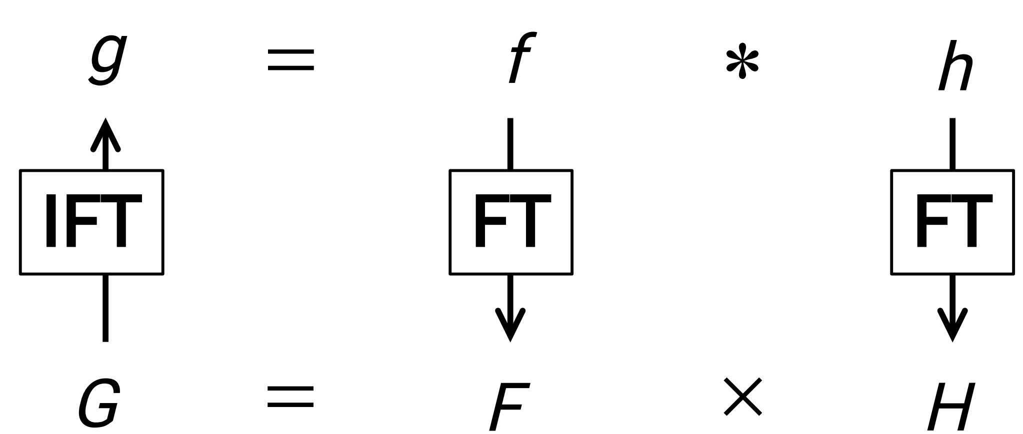
Convolution in spatial domain

$\Leftrightarrow$ Multiplication in frequency domain

Fourier Transform and Convolution

Spatial Domain *(x)*		Frequency Domain *(u)*
$g = f * h$	$\longleftrightarrow$	$G = FH$
$g = fh$	$\longleftrightarrow$	$G = F * H$

So, we can find $g(x)$ by Fourier transform



Properties of Fourier Transform

	Spatial Domain *(x)*	Frequency Domain *(u)*
Linearity	$c_1 f(x) + c_2 g(x)$	$c_1 F(u) + c_2 G(u)$
Scaling	$f(ax)$	$\frac{1}{\lvert a \rvert} F\left(\frac{u}{a}\right)$
Shifting	$f(x - x_0)$	$e^{-i2\pi u x_0} F(u)$
Symmetry	$F(x)$	$f(-u)$
Conjugation	$f^*(x)$	$F^*(-u)$
Convolution	$f(x) * g(x)$	$F(u)G(u)$
Differentiation	$\frac{d^n f(x)}{dx^n}$	$(i2\pi u)^n F(u)$

Note that these are derived using frequency ($e^{-i2\pi ux}$)

Properties of Fourier Transform

Parseval's theorem:

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(\xi)|^2 d\xi$$

$$\int_{-\infty}^{\infty} f(x) g^*(x)\, dx = \int_{-\infty}^{\infty} F(\xi) G^*(\xi)\, d\xi$$

$f(x)$	$F(\xi)$
Real(R)	Real part even (RE) Imaginary part odd (IO)
Imaginary (I)	RO,IE
RE,IO	R
RE,IE	I
RE	RE
RO	IO
IE	IE
IO	RO
Complex even (CE)	CE
CO	CO