$$f(z) = u(x, y) + iv(x, y) \quad \text{for} \quad z = x + iy$$

$$f'(z) = \lim_{\Delta z \to 0} \left[\frac{f(z+\Delta z) - f(z)}{\Delta z} \right] \quad \text{exists}$$

Its value does not depend on the direction.

Ex : Show that the function $f(z) = x^2 - y^2 + i2xy$ is differentiable for all values of z.

for $\Delta z = \Delta x + i\Delta y$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \frac{(x+\Delta x)^2 - (y+\Delta y)^2 + 2i(x+\Delta x)(y+\Delta y) - x^2 + y^2 - 2ixy}{\Delta x + i\Delta y}$$

$$= 2x + i2y + \frac{(\Delta x)^2 - (\Delta y)^2 + 2i\Delta x\Delta y}{\Delta x + i\Delta y}$$

(1) choose $\Delta y = 0, \Delta x \to 0 \Rightarrow f'(z) = 2x + i2y$

(2) choose $\Delta x = 0, \Delta y \to 0 \Rightarrow f'(z) = 2x + i2y$

** Another method :

$$f(z) = (x + iy)^2 = z^2$$

$$f'(z) = \lim_{\Delta z \to 0} \left[\frac{(z+\Delta z)^2 - z^2}{\Delta z}\right] = \lim_{\Delta z \to 0} \left[\frac{(\Delta z)^2 + 2z\Delta z}{\Delta z}\right]$$

$$= \lim_{\Delta z \to 0} \Delta z + 2z = 2z$$

Ex : Show that the function $f(z) = 2y + ix$ is not differentiable anywhere in the complex plane.

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{2y + 2\Delta y + ix + i\Delta x - 2y - ix}{\Delta x + i\Delta y} = \frac{2\Delta y + i\Delta x}{\Delta x + i\Delta y}$$

if $\Delta z \to 0$ along a line thriugh z of slope $m \Rightarrow \Delta y = m\Delta x$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta x, \Delta y \to 0} \left[\frac{2\Delta y + i\Delta x}{\Delta x + i\Delta y}\right] = \frac{2m + i}{1 + im}$$

The limit depends on m (the direction) , so $f(z)$ is nowhere differentiable.

Ex : Show that the function $f(z)=1/(1-z)$ is analytic everywhere except at $z=1$.

$$f'(z)=\lim_{\Delta z\to 0}\left[\frac{f(z+\Delta z)-f(z)}{\Delta z}\right]=\lim_{\Delta z\to 0}\left[\frac{1}{\Delta z}\left(\frac{1}{1-z-\Delta z}-\frac{1}{1-z}\right)\right]$$

$$=\lim_{\Delta z\to 0}\left[\frac{1}{(1-z-\Delta z)(1-z)}\right]=\frac{1}{(1-z)^2}$$

Provided $z\neq 1$, $f(z)$ is analytic everywhere such that $f'(z)$ is independent of the direction.

Cauchy-Riemann relation

A function f(z)=u(x,y)+iv(x,y) is differentiable and analytic, there must be particular connection between u(x,y) and v(x,y)

$$L = \lim_{\Delta z \to 0} \left[\frac{f(z+\Delta z) - f(z)}{\Delta z} \right]$$

$$f(z) = u(x, y) + iv(x, y) \qquad \Delta z = \Delta x + i\Delta y$$

$$f(z+\Delta z) = u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)$$

$$\Rightarrow L = \lim_{\Delta x, \Delta y \to 0} \left[\frac{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) - u(x, y) - iv(x, y)}{\Delta x + i\Delta y} \right]$$

(1) if suppose Δz is real $\Rightarrow \Delta y = 0$

$$\Rightarrow L = \lim_{\Delta x \to 0} \left[\frac{u(x+\Delta x, y) - u(x, y)}{\Delta x} + i\frac{v(x+\Delta x, y) - v(x, y)}{\Delta x} \right] = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

(2) if suppose Δz is imaginary $\Rightarrow \Delta x = 0$

$$\Rightarrow L = \lim_{\Delta y \to 0} \left[\frac{u(x, y+\Delta y) - u(x, y)}{i\Delta y} + i\frac{v(x, y+\Delta y) - v(x, y)}{i\Delta y} \right] = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \qquad \text{Cauchy - Riemann relations}$$

Ex : In which domain of the complex plane is

$f(z) = |x| - i|y|$ an analytic function?

$u(x, y) = |x|, \ v(x, y) = -|y|$

(1) $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \Rightarrow \dfrac{\partial}{\partial x}|x| = \dfrac{\partial}{\partial y}[-|y|] \Rightarrow$ (a) $x > 0, \ y < 0$ the fouth quatrant

(b) $x < 0, y > 0$ the second quatrant

(2) $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} \Rightarrow \dfrac{\partial}{\partial x}[-|y|] = -\dfrac{\partial}{\partial y}|x|$

$z = x + iy$ and complex conjugate of z is $z^* = x - iy$

$\Rightarrow x = (z + z^*)/2$ and $y = (z - z^*)/2i$

$$\Rightarrow \frac{\partial f}{\partial z^*} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z^*} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z^*} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

If $f(z)$ is analytic , then the Cauchy - Riemann relations

are satisfied. $\Rightarrow \partial f / \partial z^* = 0$ implies an analytic fonction of z contains

the combinatio n of $x + iy$, not $x - iy$

If Cauchy - Riemann relations are satisfied

(1) $\frac{\partial}{\partial x}(\frac{\partial u}{\partial x}) = \frac{\partial}{\partial x}(\frac{\partial v}{\partial y}) = \frac{\partial}{\partial y}(\frac{\partial v}{\partial x}) = -\frac{\partial}{\partial y}(\frac{\partial u}{\partial y}) \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial^2 y^2} = 0$

(2) the same result for function $v(x, y) \Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial^2 y^2} = 0$

$\Rightarrow u(x, y)$ and $v(x, y)$ are solutions of Laplace' s equation in two dimension.

For two families of curves $u(x, y) =$ conctant and $v(x, y) =$ constant, the normal vectors correspond ing the two curves, respective ly, are

$\vec{\nabla} u(x, y) = \frac{\partial u}{\partial x}\hat{i} + \frac{\partial u}{\partial y}\hat{j}$ and $\vec{\nabla} v(x, y) = \frac{\partial v}{\partial x}\hat{i} + \frac{\partial v}{\partial y}\hat{j}$

$\vec{\nabla} u \cdot \vec{\nabla} v = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} = 0$ orthogonal

Power series in a complex variable

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n r^n \exp(in\theta)$$

if $\sum_{n=0}^{\infty} |a_n| r^n$ is convergent $\Rightarrow f(z)$ is absolutely convergent

Is $\sum_{n=0}^{\infty} |a_n| r^n$ convergent or not, can be justified by" Cauchy root test".

The radius of convergence $R \Rightarrow \frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} \Rightarrow$ (1) $|z| < R$ absolutely convergent

(2) $|z| > R$ divergent

(3) $|z| = R$ undetermined

(1) $\sum_{n=0}^{\infty} \frac{z^n}{n!} \Rightarrow \lim_{n\to\infty} (\frac{1}{n!})^{1/n} = 0 \Rightarrow R = \infty$ converges for all z

(2) $\sum_{n=0}^{\infty} n!\, z^n \Rightarrow \lim_{n\to\infty} (n!)^{1/n} = \infty \Rightarrow R = 0$ converges only at $z = 0$

Some elementary functions

Define $\exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

Ex : Show that $\exp z_1 \exp z_2 = \exp(z_1 + z_2)$

$$\exp(z_1 + z_2) = \sum_{n=0}^{\infty} \frac{(z_1 + z_2)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}(C_0^n z_1^n + C_1^n z_1^{n-1} z_2 + C_2^n z_1^{n-2} z_2^2 + C_r^n z_1^{n-r} z_2^r + \ldots + C_n^n z_2^n)$$

set $n = r + s \Rightarrow$ the coeff. of $z_1^s z_2^r$ is $\frac{C_r^n}{n!} = \frac{1}{n!} \frac{n!}{(n-r)!\, r!} = \frac{1}{s!\, r!}$

$$\exp z_1 \exp z_2 = \sum_{s=0}^{\infty} \frac{z^s}{s!} \sum_{r=0}^{\infty} \frac{z^r}{r!} = \sum_{s=0}^{\infty} \sum_{r=0}^{\infty} \frac{1}{s!\, r!} z_1^s z_2^r$$

There are the same coeff. of $z_1^s z_2^r$ for the above two terms.

Define the complex comonent of a real number $a > 0$

$$a^z = \exp(z \ln a) = \sum_{n=0}^{\infty} \frac{z^n (\ln a)^n}{n!}$$

(1) if $a = e \Rightarrow e^z = \exp(z \ln e) = \exp z$ the same as real number

(2) if $a = e, z = iy \Rightarrow e^{iy} = \exp(iy) = 1 - \frac{y^2}{2!} - \frac{iy^3}{3!} + \ldots$

$$= 1 - \frac{y^2}{2!} + \frac{y^4}{4!} + \ldots\ldots + i(y - \frac{y^3}{3!} + \ldots) = \cos y + i \sin y$$

(3) if $a = e, z = x + iy \Rightarrow e^{x+iy} = e^x e^{iy} = \exp(x)(\cos y + i \sin y)$

Set $\exp w = z$

Write $z = r \exp i\theta$ for r is real and $-\pi < \theta \le \pi$

$\Rightarrow z = r \exp[i(\theta + 2n\pi)] \Rightarrow w = Lnz = \ln r + i(\theta + 2n\pi)$

Lnz is a multivalued function of z.

Take its principal value by choosing $n = 0$

$\Rightarrow \ln z = \ln r + i\theta \quad -\pi < \theta \le \pi$

If $t \neq 0$ and z are both complex numbers, we define

$$t^z = \exp(zLnt)$$

Ex : Show that there are exactly n distinct nth roots of t.

$$t^{\frac{1}{n}} = \exp\left(\frac{1}{n} Lnt\right) \quad \text{and} \quad t = r \exp[i(\theta + 2k\pi)]$$

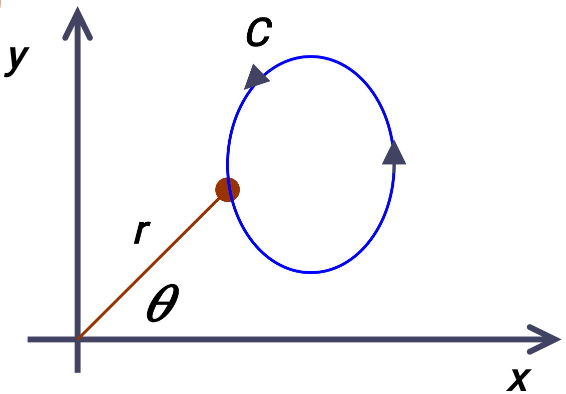
$$\Rightarrow \quad t^{\frac{1}{n}} = \exp\left[\frac{1}{n}\ln r + i\frac{(\theta + 2k\pi)}{n}\right] = r^{\frac{1}{n}} \exp\left[i\frac{(\theta + 2k\pi)}{n}\right]$$

Multivalued functions and branch cuts

A logarithmic function, a complex power and a complex root are all multivalued. Is the properties of analytic function still applied?

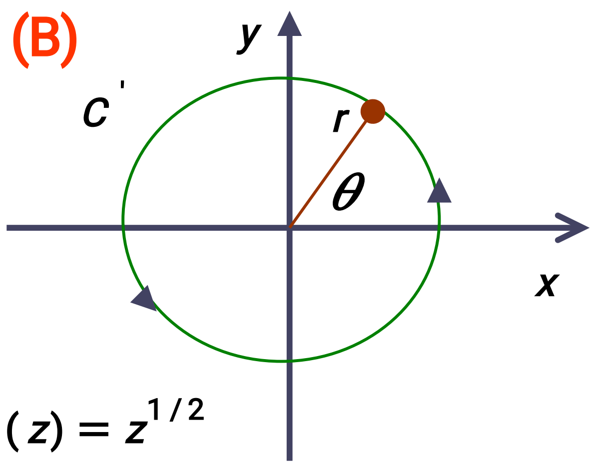
Ex : $f(z) = z^{1/2}$ and $z = r\exp(i\theta)$

(A)



(A) z traverse any closed contour C that dose not enclose the origin, θ return to its original value after one complete circuit.

(B)



(B) $\theta \to \theta + 2\pi$ enclose the origin

$$r^{1/2}\exp(i\theta/2) \to r^{1/2}\exp[i(\theta + 2\pi)/2]$$

$$= -r^{1/2}\exp(i\theta/2)$$

$$\Rightarrow f(z) \to -f(z)$$

$z = 0$ is a branch point of the function $f(z) = z^{1/2}$

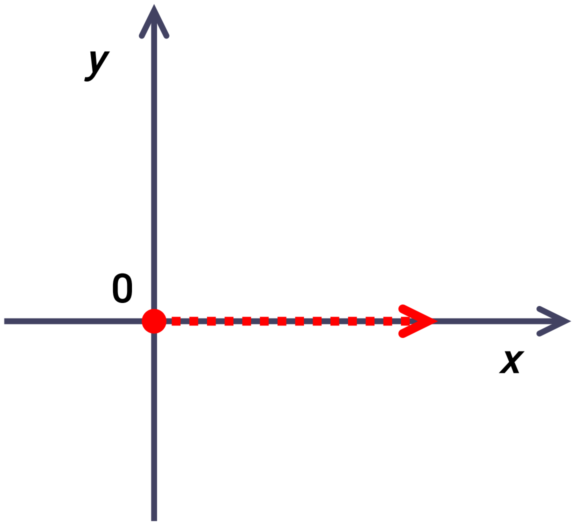
Branch point: z remains unchanged while z traverse a closed contour C about some point. But a function f(z) changes after one complete circuit.

Branch cut: It is a line (or curve) in the complex plane that we must cross , so the function remains single-valued.

Ex : $f(z) = z^{1/2}$

restrict $\theta \Rightarrow 0 \le \theta < 2\pi$

$\Rightarrow f(z)$ is single - valued



Ex : Find the branch points of $f(z)=\sqrt{z^2+1}$, and hence sketch suitable arrangements of branch cuts.

$f(z)=\sqrt{z^2+1}=\sqrt{(z+i)(z-i)}$ expected branch points : $z=\pm i$

set $z-i=r_1\exp(i\theta_1)$ and $z+i=r_2\exp(i\theta_2)$

$\Rightarrow f(z)=\sqrt{r_1 r_2}\exp(i\theta_1/2)\exp(i\theta_2/2)$

$=\sqrt{r_1 r_2}\exp[i(\theta_1+\theta_2)]$

If contour C encloses

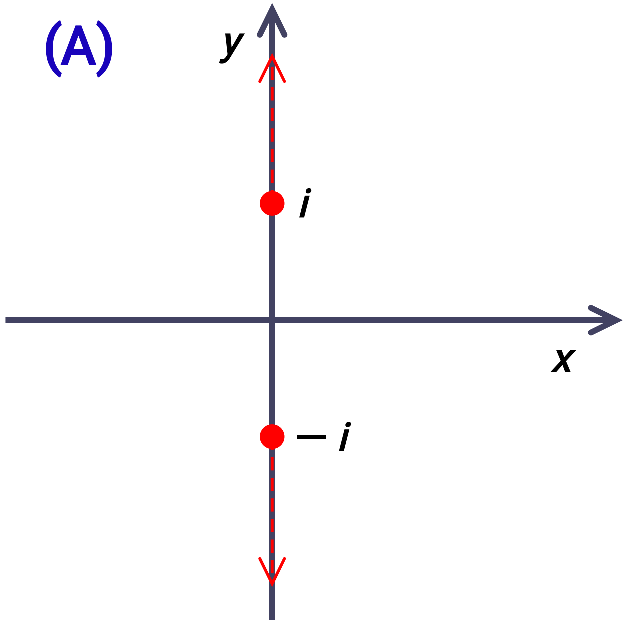
(1) neither branch point, then $\theta_1\to\theta_1, \theta_2\to\theta_2 \Rightarrow f(z)\to f(z)$

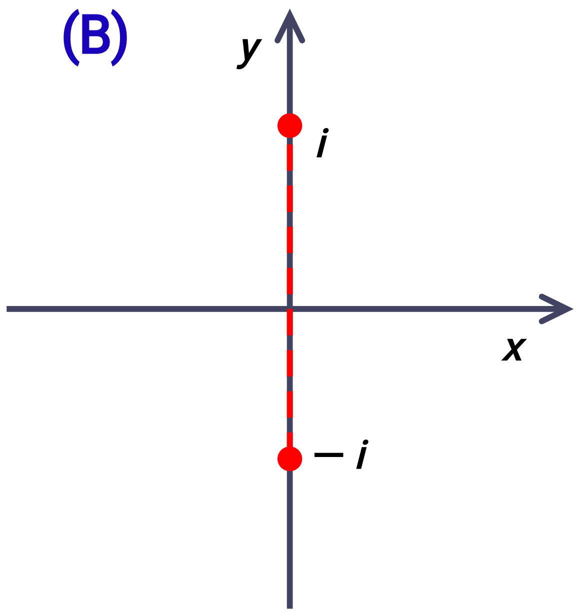
(2) $z=i$ but not $z=-i$, then $\theta_1\to\theta_1+2\pi, \theta_2\to\theta_2 \Rightarrow f(z)\to -f(z)$

(3) $z=-i$ but not $z=i$, then $\theta_1\to\theta_1, \theta_2\to\theta_2+2\pi \Rightarrow f(z)\to -f(z)$

(4) both branch points, then $\theta_1\to\theta_1+2\pi, \theta_2\to\theta_2+2\pi \Rightarrow f(z)\to f(z)$

$f(z)$ changes value around loops containing either $z = i$ or $z = -i$. We choose branch cut as follows:





Singularities and zeros of complex function

Isolated singularity (pole) : $f(z) = \dfrac{g(z)}{(z - z_0)^n}$

n is a positive integer, $g(z)$ is analytic at all points in some neighborhood containing $z = z_0$ and $g(z_0) \neq 0$, the $f(z)$ has a pole of order n at $z = z_0$.

* * An alternate definition for that $f(z)$ has a pole of order n at $z = z_0$ is

$$\lim_{z \to z_0} [(z - z_0)^n f(z)] = a$$

$f(z)$ is analytic and a is a finite, non-zero complex number

(1) if $a = 0$, then $z = z_0$ is a pole of order less than n.

(2) if a is infinite, then $z = z_0$ is a pole of order greater than n.

(3) if $z = z_0$ is a pole of $f(z) \Rightarrow |f(z)| \to \infty$ as $z \to z_0$

(4) from any direction, if no finite n satisfies the limit $\Rightarrow$ essential singularity

Ex : Find the singularities of the function

(1) $f(z) = \frac{1}{1-z} - \frac{1}{1+z}$

$\Rightarrow f(z) = \frac{2z}{(1-z)(1+z)}$ poles of order 1 at $z = 1$ and $z = -1$

(2) $f(z) = \tanh z$

$$= \frac{\sinh z}{\cosh z} = \frac{\exp z - \exp(-z)}{\exp z + \exp(-z)}$$

$f(z)$ has a singularity when $\exp z = -\exp(-z)$

$\Rightarrow \exp z = \exp[i(2n+1)\pi] = \exp(-z)$ n is any integer

$$\Rightarrow 2z = i(2n+1)\pi \Rightarrow z = (n + \frac{1}{2})\pi i$$

Using l' Hospital's rule

$$\lim_{z \to (n+1/2)\pi i} \{\frac{[z-(n+1/2)\pi i]\sinh z}{\cosh z}\} = \lim_{z \to (n+1/2)\pi i} \{\frac{[z-(n+1/2)\pi i]\cosh z + \sinh z}{\sinh z}\} = 1$$

each singularity is a simple pole (n = 1)

Remove singularities:

Singularity makes the value of $f(z)$ undetermined, but $\lim_{z \to z_0} f(z)$ exists and independent of the direction from which z_0 is approached.

Ex : Show that $f(z) = \sin z / z$ is a removable singularity at $z = 0$

Sol : $\lim_{z \to 0} f(z) = 0/0$ undetermined

$$f(z) = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \text{........}\right) = 1 - \frac{z^3}{3!} + \frac{z^5}{5!} - \text{....}$$

$\lim_{z \to 0} f(z) = 1$ is independent of the way $z \to 0$, so

$f(z)$ has a removable singularity at $z = 0$.

The behavior of $f(z)$ at infinity is given by that of $f(1/\xi)$ at $\xi = 0$, where $\xi = 1/z$

Ex : Find the behavior at infinity of (i) $f(z) = a + bz^{-2}$

(ii) $f(z) = z(1+z^2)$ and (iii) $f(z) = \exp z$

(i) $f(z) = a + bz^{-2} \Rightarrow$ set $z = 1/\xi \Rightarrow f(1/\xi) = a + b\xi^2$

is analytic at $\xi = 0 \Rightarrow f(z)$ is analytic at $z = \infty$

(ii) $f(z) = z(1 - z^2) \Rightarrow f(1/\xi) = 1/\xi + 1/\xi^3$ has a pole of order 3 at $z = \infty$

(iii) $f(z) = \exp z \Rightarrow f(1/\xi) = \sum_{n=0}^{\infty} (n!)^{-1} \xi^{-n}$

$f(z)$ has an essential singularity at $z = \infty$

If $f(z_0) = 0$ and $f(z) = (z - z_0)^n g(z)$, if n is a positive integer, and $g(z_0) \neq 0$

(i) $z = z_0$ is called a zero of order n.

(ii) if $n = 1$, $z = z_0$ is called a simple zero.

(iii) $z = z_0$ is also a pole of order n of $1/f(z)$

Complex integral

A real continuous parameter t, for $\alpha \le t \le \beta$

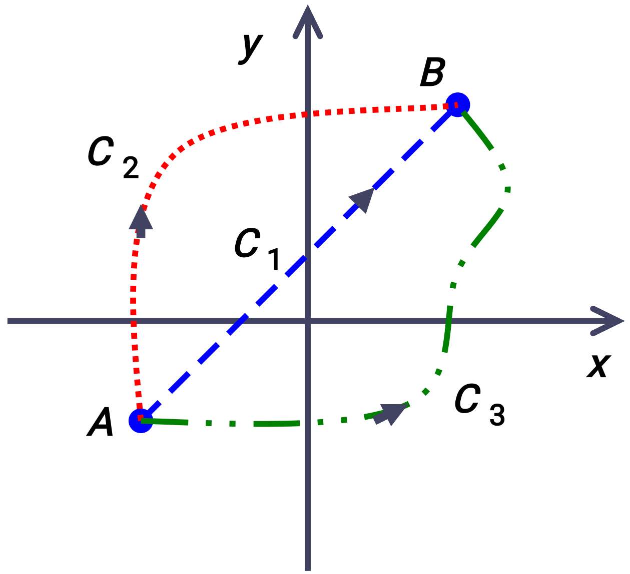
$x = x(t)$, $y = y(t)$ and point A is $t = \alpha$,

point B is $t = \beta$

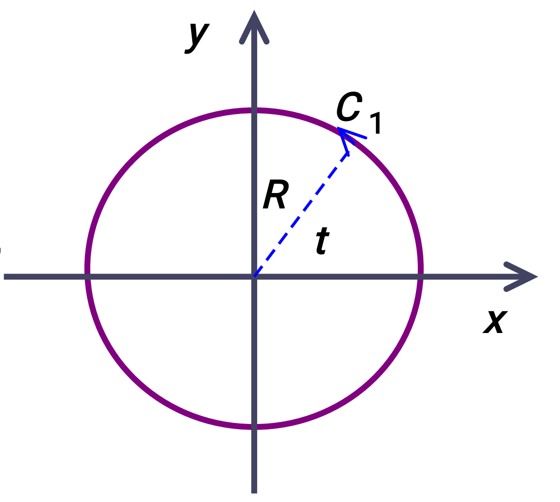
$$\Rightarrow \int_C f(z)dz = \int_C (u + iv)(dx + idy)$$

$$= \int_C udx - \int_C vdy + i\int_C udy + i\int_C vdx$$

$$= \int_\alpha^\beta u\frac{dx}{dt}dt - \int_\alpha^\beta v\frac{dy}{dt}dt + i\int_\alpha^\beta u\frac{dy}{dt}dt + i\int_\alpha^\beta v\frac{dx}{dt}dt$$



Ex : Evaluate the complex integral of $f(z) = 1/z$, along the circle $|z| = R$, starting and finishing at $z = R$.



$$z(t) = R\cos t + iR\sin t,\ 0 \le t \le 2\pi$$

$$\frac{dx}{dt} = -R\sin t,\ \frac{dy}{dt} = R\cos t,\ f(z) = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = u + iv,$$

$$u = \frac{x}{x^2+y^2} = \frac{\cos t}{R},\ v = \frac{-y}{x^2+y^2} = \frac{-\sin t}{R}$$

$$\int_{C_1} \frac{1}{z}dz = \int_0^{2\pi} \frac{\cos t}{R}(-R\sin t)\,dt - \int_0^{2\pi} \left(\frac{-\sin t}{R}\right) R\cos t\,dt$$

$$+ i\int_0^{2\pi} \frac{\cos t}{R} R\cos t\,dt + i\int_0^{2\pi} \left(\frac{-\sin t}{R}\right)(-R\sin t)\,dt$$

$$= 0 + 0 + i\pi + i\pi = 2\pi i$$

** The integral is also calculated by

$$\int_{C_1} \frac{dz}{z} = \int_0^{2\pi} \frac{-R\sin t + iR\cos t}{R\cos t + iR\sin t}dt = \int_0^{2\pi} i\,dt = 2\pi i$$

The calculated result is independen t of R.

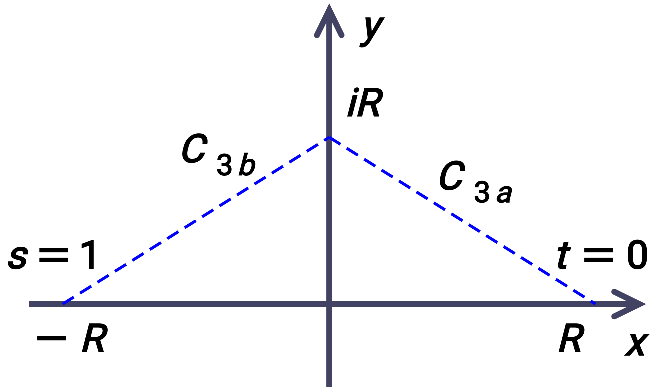
Ex : Evaluate the complex integral of $f(z) = 1/z$ along

(i) the contour C_2 consisting of the semicircle $|z| = R$ in the half-plane $y \geq 0$

(ii) the contour C_3 made up of two straight lines C_{3a} and C_{3b}

(i) This is just as in the previous example, but for

$$0 \leq t \leq \pi \Rightarrow \int_{C_2} dz/z = \pi i$$



(ii) $C_{3a} : z = (1-t)R + itR$ for $0 \leq t \leq 1$

$C_{3b} : -sR + i(1-s)R$ for $0 \leq s \leq 1$

$$\int_{C_3} \frac{dz}{z} = \int_0^1 \frac{-R + iR}{R + t(-R + iR)} dt + \int_0^1 \frac{-R - iR}{iR + s(-R - iR)} dt$$

1st term $\Rightarrow \int_0^1 \frac{-1+i}{1-t+it} dt = \int_0^1 \frac{2t-1}{1-2t+2t^2} dt + i\int_0^1 \frac{1}{1-2t+2t^2} dt$

$$= \frac{1}{2}[\ln(1-2t+2t^2)]\,|_0^1 + \frac{i}{2}[2\tan^{-1}(\frac{t-1/2}{1/2})]\,|_0^1$$

$$= 0 + \frac{i}{2}[\frac{\pi}{2} - (-\frac{\pi}{2})] = \frac{\pi i}{2}$$

$$\int \frac{a}{a^2 + x^2} dx = \tan^{-1}(\frac{x}{a}) + c$$

$$\text{2nd term} \Rightarrow \int_0^1 \frac{1+i}{s+i(s-1)}ds = \int_0^1 \frac{(1+i)[s-i(s-1)]}{s^2+(s-1)^2}ds$$

$$= \int_0^1 \frac{2s-1}{2s^2-2s+1}ds + i\int_0^1 \frac{1}{2s^2-2s+1}ds$$

$$= \frac{1}{2}[\ln(2s^2-2s+1)]\,|_0^1 + i\tan^{-1}\left(\frac{s-1/2}{1/2}\right)|_0^1$$

$$= 0 + i\left[\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)\right] = \frac{\pi i}{2}$$

$$\Rightarrow \int_{C_3} \frac{dz}{z} = \pi i$$

The integral is independent of the different path.

Ex : Evaluate the complex integral of $f(z) = \mathrm{Re}(z)$ along the path C_1, C_2 and C_3 as shown in the previous examples.

(i) C_1 : $\displaystyle\int_0^{2\pi} R\cos t(-R\sin t + iR\cos t)\,dt = i\pi R^2$

(ii) C_2 : $\displaystyle\int_0^{\pi} R\cos t(-R\sin t + iR\cos t)\,dt = \frac{i\pi}{2}R^2$

(iii) $C_3 = C_{3a} + C_{3b}$:

$$\int_0^1 (1-t)R(-R+iR)\,dt + \int_0^1 (-sR)(-R-iR)\,ds$$

$$= R^2\int_0^1 (1-t)(-1+i)\,dt + R^2\int_0^1 s(1+i)\,ds$$

$$= \frac{1}{2}R^2(-1+i) + \frac{1}{2}R^2(1+i) = iR^2$$

The integral depends on the different path.

Cauchy theorem

If $f(z)$ is an analytic function, and $f'(z)$ is continuous at each point within and on a closed contour C

$$\Rightarrow \oint_C f(z)\,dz = 0$$

If $\dfrac{\partial p(x,y)}{\partial x}$ and $\dfrac{\partial q(x,y)}{\partial y}$ are continuous within and on a closed contour C, then by two - demensional divergence theorem

$$\Rightarrow \iint_R \left(\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y}\right) dxdy = \oint_C (pdy - qdx)$$

$f(z) = u + iv$ and $dz = dx + idy$

$$I = \oint_C f(z)\,dz = \oint_C (udx - vdy) + i\oint_C (vdx + udy)$$

$$= \iint_R \left[\frac{\partial(-u)}{\partial y} + \frac{\partial(-v)}{\partial x}\right] dxdy + i\iint_R \left[\frac{\partial(-v)}{\partial y} + \frac{\partial u}{\partial x}\right] dxdy = 0$$

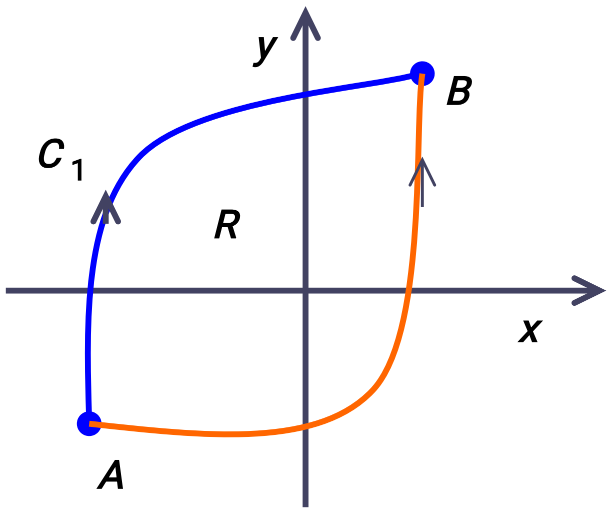
$f(z)$ is analytic and the Cauchy - Riemann relations apply.

Ex : Suppose twos points A and B in the complex plane are joined by two different paths C_1 and C_2. Show that if $f(z)$ is an analytic function on each path and in the region enclosed by the two paths then the integral of $f(z)$ is the same along C_1 and C_2.

$$\int_{C_1} f(z)dz - \int_{C_2} f(z)dz = \oint_{C_1 - C_2} f(z)dz = 0$$

path $C_1 - C_2$ forms a closed contour enclosing R

$$\Rightarrow \int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$



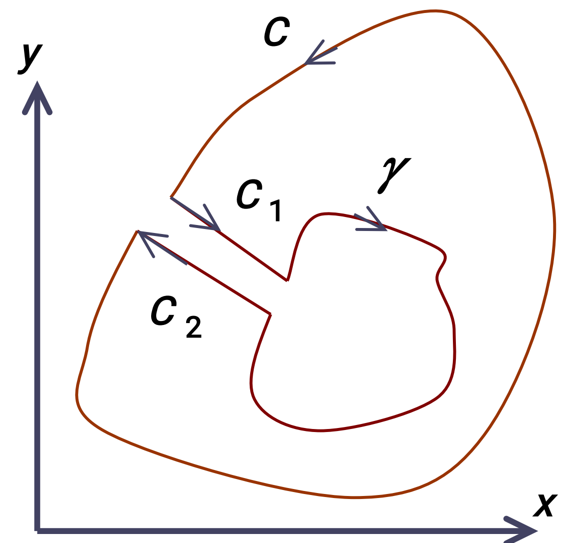
Ex : Consider two closed contour C and γ in the Argand diagram, γ being sufficiently small that it lies completely with C. Show that if the function $f(z)$ is analytic in the region between the two contours then $\oint_C f(z)dz = \oint_\gamma f(z)dz$

the area is bounded by Γ, and

$f(z)$ is analytic

$\oint_\Gamma f(z)dz = 0$

$$= \oint_C f(z)dz + \oint_\gamma f(z)dz + \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz$$



If take the direction of contour γ as that of contour $C \Rightarrow \oint_C f(z)dz = \oint_\gamma f(z)dz$

Morera's theorem :

if $f(z)$ is a continuous function of z in a closed domain R bounded by a curve C, for $\oint_C f(z)dz = 0 \Rightarrow f(z)$ is analytic.

Cauchy's integral formula

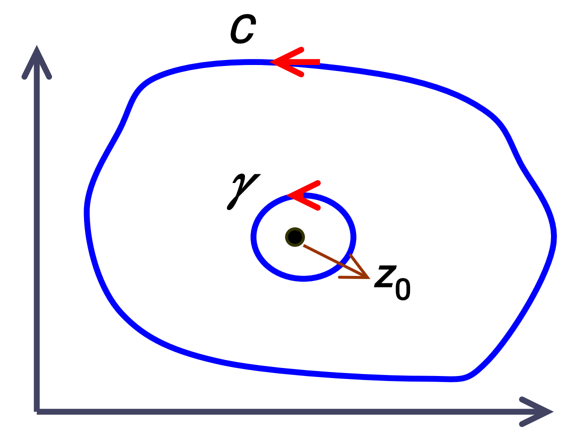
If $f(z)$ is analytic within and on a closed contour C and z_0 is a point within C then $f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\, dz$

$$I = \oint_C \frac{f(z)}{z - z_0} dz = \oint_\gamma \frac{f(z)}{z - z_0} dz$$

for $z = z_0 + \rho \exp(i\theta)$, $dz = i\rho \exp(i\theta)\, d\theta$

$$I = \int_0^{2\pi} \frac{f(z_0 + \rho e^{i\theta})}{\rho e^{i\theta}} i\rho e^{i\theta} d\theta$$

$$= i\int_0^{2\pi} f(z_0 + \rho e^{i\theta})\, d\theta \overset{\rho \to 0}{=} 2\pi i f(z_0)$$



The integral form of the derivative of a complex function :

$$f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}dz$$

$$f'(z_0) = \lim_{h\to 0}\frac{f(z_0+h) - f(z_0)}{h}$$

$$= \lim_{h\to 0}\left[\frac{1}{2\pi i}\oint_C \frac{f(z)}{h}\left(\frac{1}{z-z_0-h} - \frac{1}{z-z_0}\right)dz\right]$$

$$= \lim_{h\to 0}\left[\frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0-h)(z-z_0)}dz\right]$$

$$= \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}dz$$

For nth derivative $f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz$

Ex : Suppose that $f(z)$ is analytic inside and on a circle C of radius R centered on the point $z = z_0$. If $|f(z)| \leq M$ on the circle, where M is some constant, show that $|f^{(n)}(z_0)| \leq \dfrac{Mn!}{R^n}$.

$$| f^{(n)}(z_0) | = \frac{n!}{2\pi} \left| \oint_C \frac{f(z)\,dz}{(z - z_0)^{n+1}} \right| \leq \frac{n!}{2\pi} \frac{M}{R^{n+1}} 2\pi R = \frac{Mn!}{R^n}$$

Liouville's theorem : If $f(z)$ is analytic and bounded for all z then $f(z)$ is a constant.

Using Cauchy's inequality : $| f^{(n)}(z_0) | \leq \dfrac{Mn!}{R^n}$

set $n = 1$ and let $R \to \infty \Rightarrow | f'(z_0) | = 0 \Rightarrow f'(z_0) = 0$

Since $f(z)$ is analytic for all z, we may take z_0 as any point in the z-plane. $f'(z) = 0$ for all $z \Rightarrow f(z) = \text{constant}$

Taylor and Laurent series

Taylor's theorem:

If $f(z)$ is analytic inside and on a circle C of radius R centered on the point $z = z_0$, and z is a point inside C, then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

$f(z)$ is analytic inside and on C, so $f(z) = \frac{1}{2\pi i}\oint_C \frac{f(\xi)}{\xi - z} d\xi$ where ξ lies on C

expand $\frac{1}{\xi - z}$ as a geometric series in $\frac{z - z_0}{\xi - z_0} \Rightarrow \frac{1}{\xi - z} = \frac{1}{\xi - z_0}\sum_{n=0}^{\infty}\left(\frac{z - z_0}{\xi - z_0}\right)^n$

$$\Rightarrow f(z) = \frac{1}{2\pi i}\oint_C \frac{f(\xi)}{\xi - z_0}\sum_{n=0}^{\infty}\left(\frac{z - z_0}{\xi - z_0}\right)^n d\xi = \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z - z_0)^n \oint_C \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi$$

$$= \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z - z_0)^n \frac{2\pi i f^{(n)}(z_0)}{n!} = \sum_{n=0}^{\infty}(z - z_0)^n \frac{f^{(n)}(z_0)}{n!}$$

If $f(z)$ has a pole of order p at $z = z_0$ but is analytic at every other point inside and on C. Then $g(z) = (z - z_0)^p f(z)$ is analytic at $z = z_0$ and expanded as a Taylor series $g(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n$.

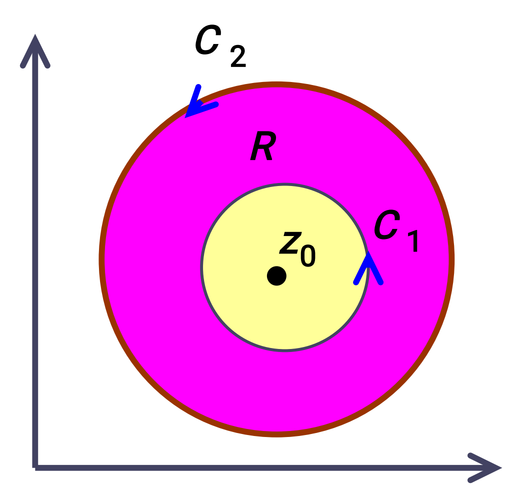
Thus, for all z inside C f(z) can be exp *anded as a Laurent s eries*

$$f(z) = \frac{a_{-p}}{(z - z_0)^p} + \frac{a_{-p+1}}{(z - z_0)^{p-1}} + \ldots\ldots + \frac{a_{-1}}{z - z_0} + a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2$$

$$a_n = b_{n+p} \quad \text{and} \quad b_n = \frac{g^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \oint \frac{g(z)}{(z - z_0)^{n+1}} dz$$

$$\Rightarrow a_n = \frac{1}{2\pi i} \oint \frac{g(z)}{(z - z_0)^{n+1+p}} dz = \frac{1}{2\pi i} \oint \frac{f(z)}{(z - z_0)^{n+1}} dz$$

$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ is analytic in a region R between two circles C_1 and C_2 centered on $z = z_0$



$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

(1) If $f(z)$ is analytic at $z = z_0$, then all $a_n = 0$ for $n < 0$.

It may happen $a_n = 0$ for $n \geq 0$, the first non - vanishing term is $a_m (z\text{-}z_0)^m$ with $m > 0$, $f(z)$ is said to have a zero of order m at $z = z_0$.

(2) If $f(z)$ is not analytic at $z = z_0$

(i) possible to find $a_{-p} \neq 0$ but $a_{-p-k} = 0$ for all $k > 0$

$f(z)$ has a pole of order p at $z = z_0$, a_{-1} is called the residue of $f(z)$

(ii) impossible to find a lowest value of $-p \Rightarrow$ essential singularit y

Ex : Find the Laurent series of $f(z)=\dfrac{1}{z(z-2)^3}$ about the singularities $z=0$ and $z=2$. Hence verify that $z=0$ is a pole of order 1 and $z=2$ is a pole of order 3, and find the residue of $f(z)$ at each pole.

(1) point $z=0$

$$f(z)=\frac{-1}{8z(1-z/2)^3}=\frac{-1}{8z}[1+(-3)(\frac{-z}{2})+\frac{(-3)(-4)}{2!}(\frac{-z}{2})^2+\frac{(-3)(-4)(-5)}{3!}(\frac{-z}{2})^3+\ldots]$$

$$=-\frac{1}{8z}-\frac{3}{16}-\frac{3}{16}z-\frac{5z^2}{32}-\ldots$$ $z=0$ is a pole of order 1

(2) point $z=2 \Rightarrow$ set $z-2=\xi \Rightarrow z(z-2)^3=(2+\xi)\xi^3=2\xi^3(1+\xi/2)$

$$f(z)=\frac{1}{2\xi^3(1+\xi/2)}=\frac{1}{2\xi^3}[1-(\frac{\xi}{2})+(\frac{\xi}{2})^2-(\frac{\xi}{2})^3+(\frac{\xi}{2})^4-\ldots]$$

$$=\frac{1}{2\xi^3}-\frac{1}{4\xi^2}+\frac{1}{8\xi}-\frac{1}{16}+\frac{\xi}{32}-..=\frac{1}{2(z-2)^3}-\frac{1}{4(z-2)^2}+\frac{1}{8(z-2)}-\frac{1}{16}+\frac{z-2}{32}-$$

$z=2$ is a pole of order 3, the residue of $f(z)$ at $z=2$ is $1/8$.

How to obtain the residue ?

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \ldots\ldots + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots$$

$$\Rightarrow (z-z_0)^m \; f(z) = a_{-m} + a_{-m+1}(z-z_0) + \ldots\ldots\ldots + a_{-1}(z-z_0)^{m-1} + \ldots$$

$$\Rightarrow \frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m \; f(z)] = (m-1)! \; a_{-1} + \sum_{n=1}^{\infty} b_n (z-z_0)^n$$

Take the limit $z \rightarrow z_0$

$$R(z_0) = a_{-1} = \lim_{z \rightarrow z_0} \{\frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m \; f(z)]\} \quad \text{residue at } z = z_0$$

(1) For a simple pole $m = 1 \Rightarrow R(z_0) = \lim_{z \rightarrow z_0} [(z-z_0) \; f(z)]$

(2) If $f(z)$ has a simple at $z = z_0$ and $f(z) = \frac{g(z)}{h(z)}$, $g(z)$ is analytic and

non - zero at z_0 and $h(z_0) = 0$

$$\Rightarrow R(z_0) = \lim_{z \rightarrow z_0} \frac{(z-z_0)\, g(z)}{h(z)} = g(z_0) \lim_{z \rightarrow z_0} \frac{(z-z_0)}{h(z)} = g(z_0) \lim_{z \rightarrow z_0} \frac{1}{h'(z)} = \frac{g(z_0)}{h'(z_0)}$$

Ex : Suppose that $f(z)$ has a pole of order m at the point $z = z_0$. By considerin g the Laurent series of $f(z)$ about z_0, deriving a general expression for the residue $R(z_0)$ of $f(z)$ at $z = z_0$. Hence evaluate the residue of the function $f(z) = \dfrac{\exp iz}{(z^2+1)^2}$ at the point $z = i$.

$$f(z) = \frac{\exp iz}{(z^2+1)^2} = \frac{\exp iz}{(z+i)^2(z-i)^2}$$ poles of order 2 at $z = i$ and $z = -i$

for pole at $z = i$:

$$\frac{d}{dz}[(z-i)^2 f(z)] = \frac{d}{dz}[\frac{\exp iz}{(z+i)^2}] = \frac{i}{(z+i)^2}\exp iz - \frac{2}{(z+i)^3}\exp iz$$

$$R(i) = \frac{1}{1!}[\frac{i}{(2i)^2}e^{-1} - \frac{2}{(2i)^3}e^{-1}] = \frac{-i}{2e}$$

Residue theorem

$f(z)$ has a pole of order m at $z = z_0$

$$f(z) = \sum_{n=-m}^{\infty} a_n (z - z_0)^n$$

$$I = \oint_C f(z)\,dz = \oint_\gamma f(z)\,dz$$

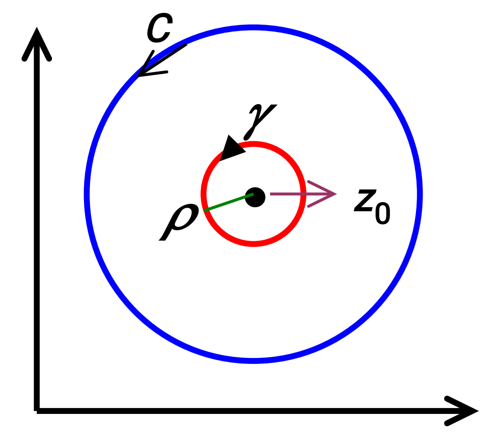
set $z = z_0 + \rho e^{i\theta} \Rightarrow dz = i\rho e^{i\theta} d\theta$

$$I = \sum_{n=-m}^{\infty} a_n \oint_C (z - z_0)^n\,dz = \sum_{n=-m}^{\infty} a_n \int_0^{2\pi} i\rho^{n+1} e^{i(n+1)\theta} d\theta$$

for $n \neq -1 \Rightarrow \int_0^{2\pi} i\rho^{n+1} e^{i(n+1)\theta} d\theta = \dfrac{i\rho^{n+1} e^{i(n+1)\theta}}{i(n+1)} \Big|_0^{2\pi} = 0$

for $n = 1 \Rightarrow \int_0^{2\pi} i\,d\theta = 2\pi i$

$$I = \oint_C f(z)\,dz = 2\pi i a_{-1}$$



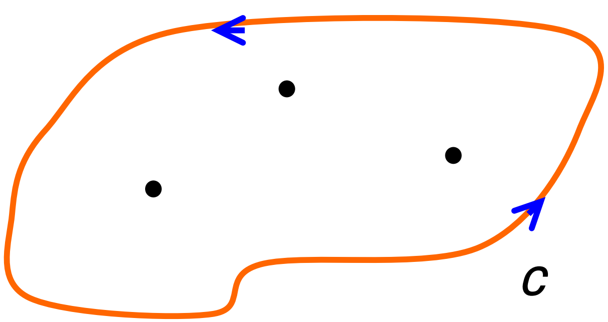
Residue theorem:

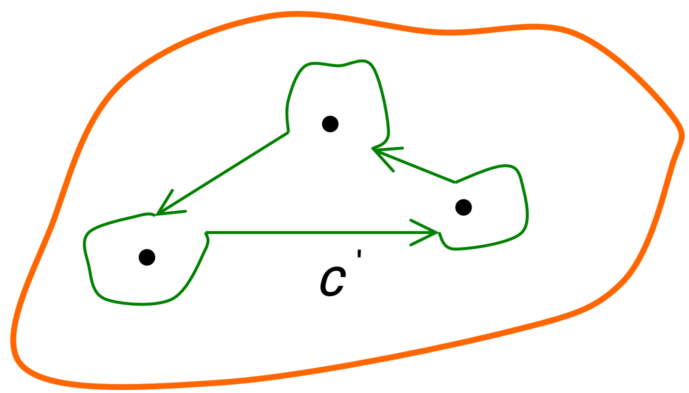
$f(z)$ is continuous within and on a closed contour C

and analytic, except for a finite number of poles within C

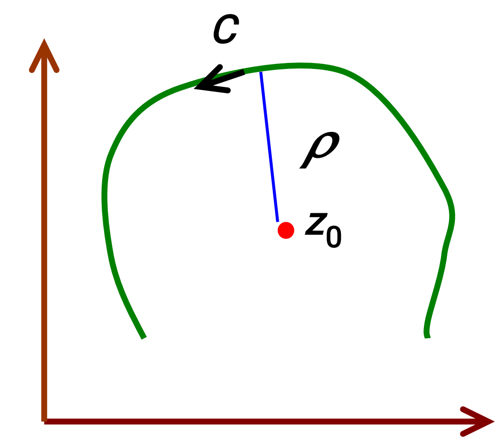
$$\oint_C f(z)\,dz = 2\pi i \sum_j R_j$$

$\sum_j R_j$ is the sum of the residues of $f(z)$ at its poles within C





The integral I of $f(z)$ along an open contour C



if $f(z)$ has a simple pole at $z = z_0$

$$\Rightarrow f(z) = \phi(z) + a_{-1}(z - z_0)^{-1}$$

$\phi(z)$ is analytic within some neighbour surroundin g z_0

$|z - z_0| = \rho$ and $\theta_1 \le \arg(z - z_0) \le \theta_2$

ρ is chosen small enough that no singularit y of $f(z)$ except $z = z_0$

$$I = \int_C f(z)\,dz = \int_C \phi(z)\,dz + a_{-1}\int_C (z - z_0)^{-1}\,dz$$

$$\lim_{\rho\to 0}\int_C \phi(z)\,dz = 0$$

$$I = \lim_{\rho\to 0}\int_C f(z)\,dz = \lim_{\rho\to 0}\left(a_{-1}\int_{\theta_1}^{\theta_2}\frac{1}{\rho e^{i\theta}} i\rho e^{i\theta}\,d\theta\right) = ia_{-1}(\theta_2 - \theta_1)$$

for a closed contour $\theta_2 = \theta_1 + 2\pi \Rightarrow I = 2\pi i a_{-1}$

Integrals of sinusoidal functions

$$\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta \quad \text{set } z = \exp i\theta \text{ in unit circle}$$

$$\Rightarrow \cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right), \quad \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right), \quad d\theta = -iz^{-1}dz$$

Ex : Evaluate $I = \int_0^{2\pi} \frac{\cos 2\theta}{a^2 + b^2 - 2ab\cos\theta} d\theta$ for $b > a > 0$

$$\cos n\theta = \frac{1}{2}(z^n + z^{-n}) \Rightarrow \cos 2\theta = \frac{1}{2}(z^2 + z^{-2})$$

$$\frac{\cos 2\theta}{a^2 + b^2 - 2ab\cos\theta} d\theta = \frac{\frac{1}{2}(z^2 + z^{-2})(-iz^{-1})dz}{a^2 + b^2 - 2ab \cdot \frac{1}{2}(z + z^{-1})} = \frac{-\frac{1}{2}(z^4 + 1)idz}{z^2(za^2 + zb^2 - abz^2 - ab)}$$

$$= \frac{i}{2ab} \frac{(z^4 + 1)dz}{z^2\left(z^2 - z\left(\frac{a}{b} - + \frac{b}{a}\right) + 1\right)} = \frac{i}{2ab} \frac{(z^4 + 1)}{z^2\left(z - \frac{a}{b}\right)\left(z - \frac{b}{a}\right)} dz$$

$$I=\frac{i}{2ab}\oint_C \frac{z^4+1}{z^2(z-\frac{a}{b})(z-\frac{b}{a})}dz$$ double poles at $z=0$ and $z=a/b$ within the unit circle

Residue : $R(z_0)=\lim_{z\to z_0}\{\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m\ f(z)]$

(1) pole at $z=0,\ m=2$

$$R(0)=\lim_{z\to 0}\{\frac{1}{1!}\frac{d}{dz}[z^2\frac{z^4+1}{z^2(z-a/b)(z-b/a)}]\}$$

$$=\lim_{z\to 0}\{\frac{4z^3}{(z-a/b)(z-b/a)}+\frac{(z^4+1)(-1)[2z-(a/b+b/a)]}{(z-a/b)^2(z-b/a)^2}\}=a/b+b/a$$

(2) pole at $z=a/b,\ m=1$

$$R(a/b)=\lim_{z\to a/b}[(z-a/b)\frac{z^4+1}{z^2(z-a/b)(z-b/a)}]=\frac{(a/b)^4+1}{(a/b)^2(a/b-b/a)}=\frac{-(a^4+b^4)}{ab(b^2-a^2)}$$

$$I=2\pi i\times\frac{i}{2ab}[\frac{a^2+b^2}{ab}-\frac{a^4+b^4}{ab(b^2-a^2)}]=\frac{2\pi a^2}{b^2(b^2-a^2)}$$

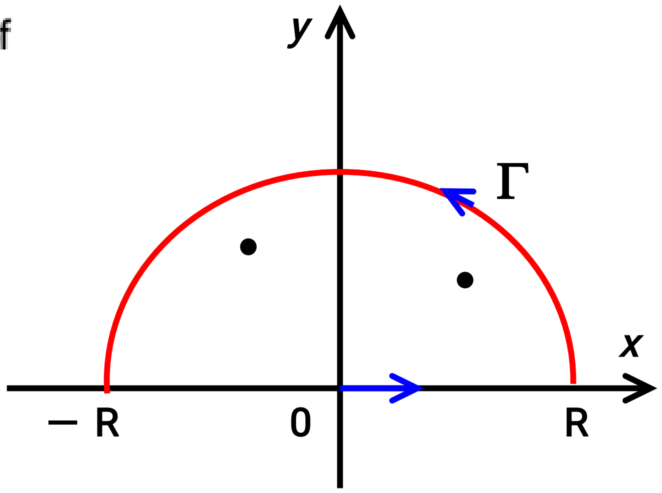
Some infinite integrals

$$\int_{-\infty}^{\infty} f(x)\,dx$$

$f(z)$ has the following properties :

(1) $f(z)$ is analytic in the upper half - plane, $\operatorname{Im} z \geq 0$, except for a finite number of poles, none of which is on the real axis.

(2) on a semicircle Γ of radius R, R times the maximum of $|f|$ on Γ tends to zero as $R \to \infty$ (a sufficient condition is that $zf(z) \to 0$ as $|z| \to \infty$).

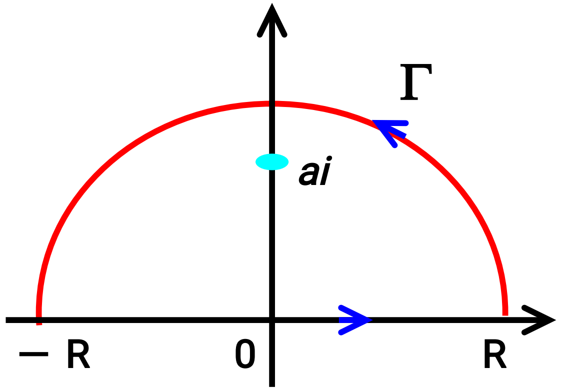


(3) $\int_{-\infty}^{0} f(x)\,dx$ and $\int_{0}^{\infty} f(x)\,dx$ both exist

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_j R_j$$

for $\left|\int_{\Gamma} f(z)\,dz\right| \leq 2\pi R \times$ (maximum of $|f|$ on Γ), the integral along Γ tends to zero as $R \to \infty$.

Ex : Evaluate $I = \int_0^\infty \frac{dx}{(x^2 + a^2)^4}$ a is real



$$\oint_C \frac{dz}{(z^2 + a^2)^4} = \int_{-R}^{R} \frac{dx}{(x^2 + a^2)^4} + \int_\Gamma \frac{dz}{(z^2 + a^2)^4} \text{ as } R \to \infty$$

$$\Rightarrow \int_\Gamma \frac{dz}{(z^2 + a^2)^4} \to 0 \Rightarrow \oint_C \frac{dz}{(z^2 + a^2)^4} = \int_{-\infty}^{\infty} \frac{dx}{(x^2 + a^2)^4}$$

$(z^2 + a^2)^4 = 0 \Rightarrow$ poles of order 4 at $z = \pm ai$,

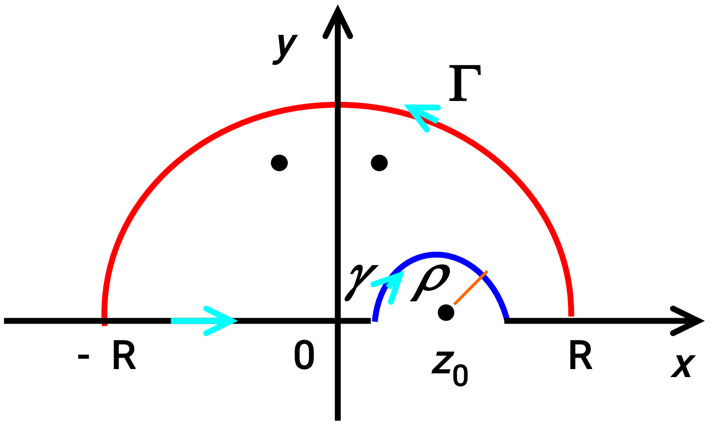
only $z = ai$ at the upper half - plane

$$\text{set } z = ai + \xi, \xi \to 0 \Rightarrow \frac{1}{(z^2 + a^2)^4} = \frac{1}{(2ai\xi + \xi^2)^4} = \frac{1}{(2ai\xi)^4}\left(1 - \frac{i\xi}{2a}\right)^{-4}$$

$$\text{the coefficient of } \xi^{-1} \text{ is } \frac{1}{(2a)^4} \frac{(-4)(-5)(-6)}{3!}\left(\frac{-i}{2a}\right)^3 = \frac{-5i}{32a^7}$$

$$\int_0^\infty \frac{dx}{(x^2 + a^2)^4} = 2\pi i\left(\frac{-5i}{32a^7}\right) = \frac{10\pi}{32a^7} \Rightarrow I = \frac{1}{2} \times \frac{10\pi}{32a^7} = \frac{5\pi}{32a^7}$$

For poles on the real axis:



Principal value of the integral, defined as $\rho \to 0$

$$P\int_{-R}^{R} f(x)dx = \int_{-R}^{z_0-\rho} f(x)dx + \int_{z_0+\rho}^{R} f(x)dx$$

for a closed contour C

$$\oint_C f(z)dz = \int_{-R}^{z_0-\rho} f(x)dx + \int_{\gamma} f(z)dz + \int_{z_0+\rho}^{R} f(x)dx + \int_{\Gamma} f(z)dz$$

$$= P\int_{-R}^{R} f(x)dx + \int_{\gamma} f(z)dz + \int_{\Gamma} f(z)dz$$

(1) for $\int_{\gamma} f(z)dz$ has a pole at $z = z_0 \Rightarrow \int_{\gamma} f(z)dz = -\pi i a_1$

(2) for $\int_{\Gamma} f(z)dz$ set $z = Re^{i\theta}$ $dz = i\,Re^{i\theta}d\theta$

$$\Rightarrow \int_{\Gamma} f(z)dz = \int_{\Gamma} f(Re^{i\theta})\,i\,Re^{i\theta}d\theta$$

If $f(z)$ vanishes faster than $1/R^2$ as $R \to \infty$, the integral is zero

Jordan's lemma

(1) $f(z)$ is analytic in the upper half-plane except for a finite number of poles in $\text{Im } z > 0$

(2) the maximum of $|f(z)| \to 0$ as $|z| \to \infty$ in the upper half-plane

(3) $m > 0$, then

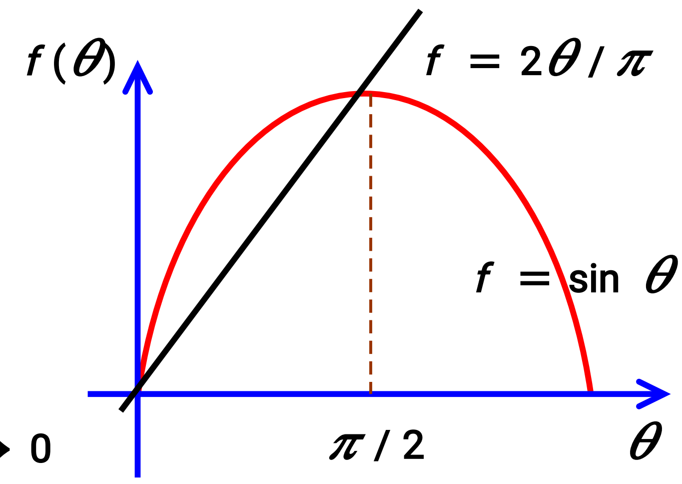
$I_\Gamma = \int_\Gamma e^{imz} f(z)\,dz \to 0$ as $R \to \infty$, Γ is the semicircular contour

for $0 \le \theta \le \pi/2$, $1 \ge \sin\theta/\theta \ge \pi/2$

$|\exp(imz)| = |\exp(-mR\sin\theta)|$

$$I_\Gamma \le \int_\Gamma |e^{imz} f(z)||dz| \le MR\int_0^\pi e^{-mR\sin\theta}\,d\theta$$

$$= 2MR\int_0^{\pi/2} e^{-mR\sin\theta}\,d\theta$$

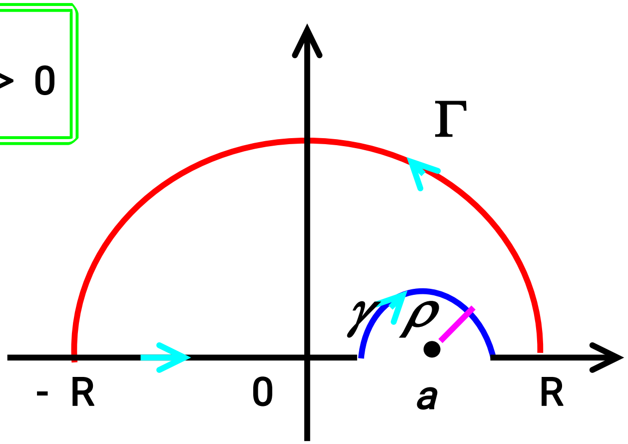


M is the maximum of $|f(z)|$ on $|z| = R$, $R \to \infty$ $M \to 0$

$$I_\Gamma < 2MR\int_0^{\pi/2} e^{-mR(2\theta/\pi)}\,d\theta = \frac{\pi M}{m}(1 - e^{-mR}) < \frac{\pi M}{m}$$

as $R \to \infty \Rightarrow M \to 0 \Rightarrow I_\Gamma \to 0$

Ex : Find the principal value of $\int_{-\infty}^{\infty} \frac{\cos mx}{x-a} dx$ a real, $m > 0$



Consider the integral $I = \oint_C \frac{e^{imz}}{z-a} dz = 0$ no pole in the upper half - plane, and $|(z-a)^{-1}| \to 0$ as $|z| \to \infty$

$$I = \oint_C \frac{e^{imz}}{z-a} dz$$

$$= \int_{-R}^{a-\rho} \frac{e^{imx}}{x-a} dx + \oint_\gamma \frac{e^{imz}}{z-a} dz + \int_{a+\rho}^{R} \frac{e^{imx}}{x-a} dx + \int_\Gamma \frac{e^{imz}}{z-a} dz = 0$$

As $R \to \infty$ and $\rho \to 0 \Rightarrow \int_\Gamma \frac{e^{imz}}{z-a} dz \to 0$

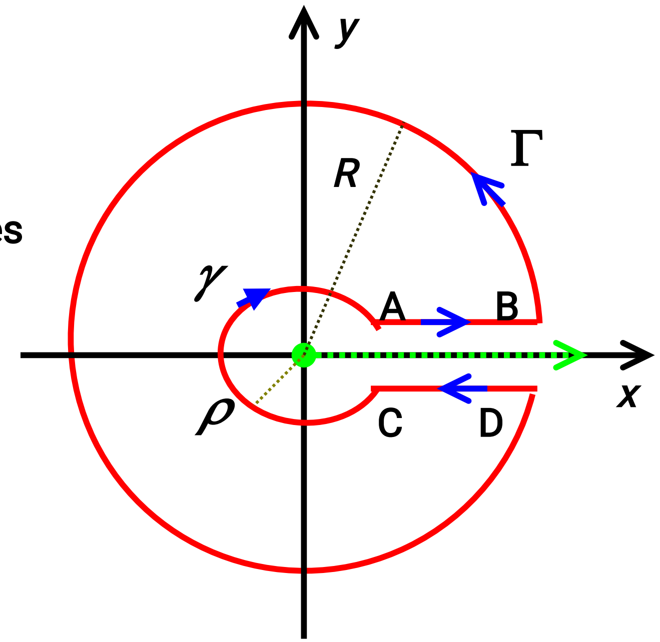
$$\Rightarrow P\int_{-\infty}^{\infty} \frac{e^{imx}}{x-a} dx - i\pi a_{-1} = 0 \text{ and } a_{-1} = e^{ima}$$

$$\Rightarrow P\int_{-\infty}^{\infty} \frac{\cos mx}{x-a} dx = -\pi \sin ma \text{ and } P\int_{-\infty}^{\infty} \frac{\sin mx}{x-a} dx = \pi \cos ma$$

Integral of multivalued functions

Multivalued functions such as $z^{1/2}$, Lnz

Single branch point is at the otigin. We let $R \to \infty$ and $\rho \to 0$. The integrand is multivalued, its values along two lines AB and CD joining $z = \rho$ to $z = R$ are not equal and opposite.



$$\text{Ex} : I = \int_0^\infty \frac{dx}{(x+a)^3 x^{1/2}} \quad \text{for } a > 0$$

(1) the integrand $f(z) = (z+a)^{-3} z^{-1/2}$, $|zf(z)| \to 0$ as $\rho \to 0$ and $R \to \infty$

the two circles make no contribution to the contour integral

(2) pole at $z = -a$, and $(-a)^{1/2} = a^{1/2} e^{i\pi/2} = ia^{1/2}$

$$R(-a) = \lim_{z \to -a} \frac{1}{(3-1)!} \frac{d^{3-1}}{dz^{3-1}} [(z+a)^3 \frac{1}{(z+a)^3 z^{1/2}}]$$

$$= \lim_{z \to -a} \frac{1}{2!} \frac{d^2}{dz^2} z^{-1/2} = \frac{-3i}{8a^{5/2}}$$

$$\int_{AB} dz + \int_{\Gamma} dz + \int_{DC} dz + \int_{\gamma} dz = 2\pi i\left(\frac{-3i}{8a^{5/2}}\right)$$

and $\int_{\Gamma} dz = 0$ and $\int_{\gamma} dz = 0$

along line AB $\Rightarrow z = xe^{i0}$, along line CD $\Rightarrow z = xe^{i2\pi}$

$$\int_{0,\, A\to B}^{\infty} \frac{dx}{(x+a)^3 x^{1/2}} + \int_{\infty,\, C\to D}^{0} \frac{dx}{(xe^{i2\pi}+a)^3 x^{1/2} e^{(1/2\times 2\pi i)}} = \frac{3\pi}{4a^{5/2}}$$

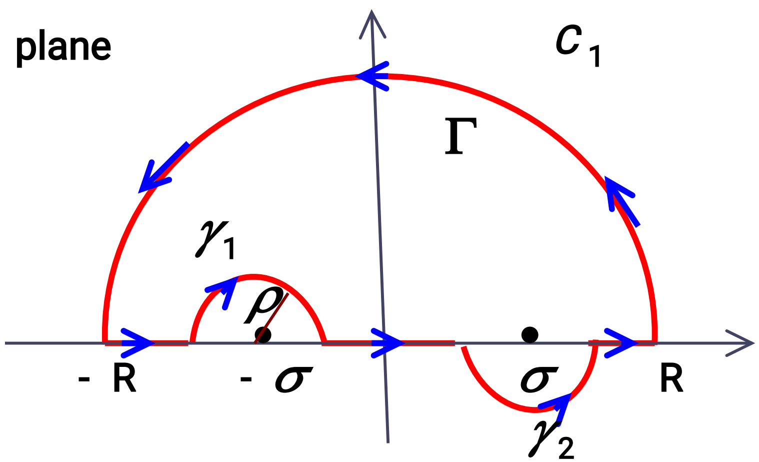
$$\Rightarrow \left(1 - \frac{1}{e^{i\pi}}\right)\int_{0}^{\infty} \frac{dx}{(x+a)^3 x^{1/2}} = \frac{3\pi}{4a^{5/2}}$$

$$\Rightarrow \int_{0}^{\infty} \frac{dx}{(x+a)^3 x^{1/2}} = \frac{3\pi}{8a^{5/2}}$$

Ex : Evaluate $I(\sigma) = \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 - \sigma^2} dx$

$$\oint_C \frac{z \sin z}{z^2 - \sigma^2} dz = \frac{1}{2i} \oint_{C_1} \frac{ze^{iz}}{z^2 - \sigma^2} dz - \frac{1}{2i} \oint_{C_2} \frac{ze^{-iz}}{z^2 - \sigma^2} dz = I_1 + I_2$$

(1) for I_1, the contour is choosed on the upper half - plane due to the term e^{iz}, and only one pole at $z = \sigma$.



$$I_1 = \frac{1}{2i} \oint_{C_1} \frac{ze^{iz}}{z^2 - \sigma^2} dz = \frac{1}{2i} \int_{-R}^{-\sigma-\rho} \frac{xe^{ix}}{x^2 - \sigma^2} dx$$

$$+ \frac{1}{2i} \int_{-\sigma+\rho}^{\sigma-\rho} \frac{xe^{ix}}{x^2 - \sigma^2} dx + \frac{1}{2i} \int_{\sigma+\rho}^{\infty} \frac{xe^{ix}}{x^2 - \sigma^2} dx$$

$$+ \frac{1}{2i} \int_{\gamma_1} \frac{ze^{iz}}{z^2 - \sigma^2} dz + \frac{1}{2i} \int_{\gamma_2} \frac{ze^{iz}}{z^2 - \sigma^2} dz + \frac{1}{2i} \int_{\Gamma} \frac{ze^{iz}}{z^2 - \sigma^2} dz$$

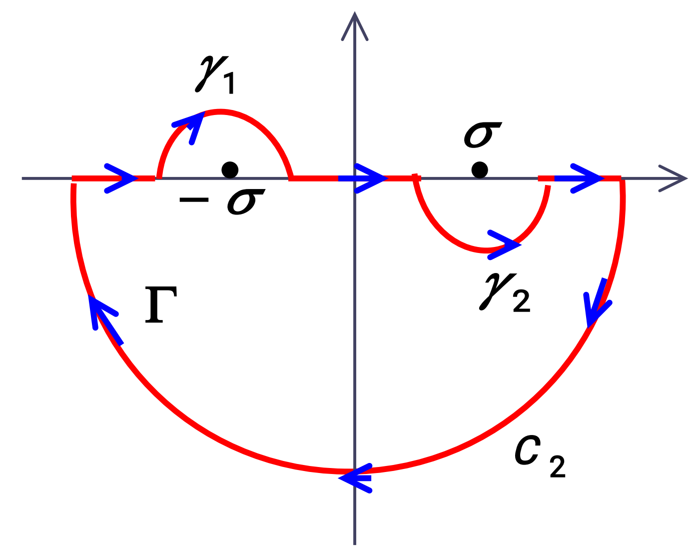
$$= \frac{1}{2i} 2\pi i \times \text{Res}(z = \sigma) = \pi \frac{\sigma e^{i\sigma}}{2\sigma} = \frac{\pi}{2} e^{i\sigma}$$

As $\rho \to 0$ and $R \to \infty \Rightarrow \int_{\Gamma} dz \to 0$

$$\frac{1}{2i}\int_{\gamma_1} \frac{ze^{iz}}{(z+\sigma)(z-\sigma)}dz = \frac{1}{2i}\times(-\pi i)\mathrm{Res}(\ z=-\sigma) = \frac{-\pi}{4}e^{-i\sigma}$$

$$\frac{1}{2i}\int_{\gamma_2} \frac{ze^{iz}}{(z+\sigma)(z-\sigma)}dz = \frac{1}{2i}\times \pi i\mathrm{Res}(\ z=\sigma) = \frac{\pi}{4}e^{i\sigma}$$

$$I_1 = \frac{1}{2i}\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2-\sigma^2}dx + \frac{\pi}{4}(e^{i\sigma}-e^{-i\sigma}) = \frac{\pi}{2}e^{i\sigma}$$



(2) for I_2, we choose the lower half - plane by the

term e^{-iz}, only one pole at $z=-\sigma$

$$I_2 = \frac{-1}{2i}\oint_{C_2}\frac{ze^{-iz}}{z^2-\sigma^2}dz = \frac{-1}{2i}\int_{-R}^{-\sigma-\rho}\frac{xe^{-ix}}{x^2-\sigma^2}dx$$

$$-\frac{1}{2i}\int_{-\sigma+\rho}^{\sigma-\rho}\frac{xe^{-ix}}{x^2-\sigma^2}dx - \frac{1}{2i}\int_{\sigma+\rho}^{\infty}\frac{xe^{-ix}}{x^2-\sigma^2}dx - \frac{1}{2i}\int_{\gamma_1}\frac{ze^{-iz}}{z^2-\sigma^2}dz$$

$$-\frac{1}{2i}\int_{\gamma_2}\frac{ze^{-iz}}{z^2-\sigma^2}dz - \frac{1}{2i}\int_{\Gamma}\frac{ze^{-iz}}{z^2-\sigma^2}dz = \left(\frac{-1}{2i}\right)\times(-2\pi i)\frac{(-\sigma)e^{i\sigma}}{-2\sigma} = \frac{\pi}{2}e^{i\sigma}$$

As $\rho \to 0,\ R \to \infty \Rightarrow \int_{\Gamma} dz \to 0$

$$\frac{-1}{2i}\int_{\gamma_1} \frac{ze^{-iz}}{(z+\sigma)(z-\sigma)}dz = (\frac{-1}{2i})(-\pi i)\ \frac{(-\sigma)e^{i\sigma}}{-2\sigma} = \frac{\pi}{4}\ e^{i\sigma}$$

$$\frac{-1}{2i}\int_{\gamma_2} \frac{ze^{-iz}}{(z+\sigma)(z-\sigma)}dz = (\frac{-1}{2i})(\pi i)\ \frac{\sigma e^{-i\sigma}}{2\sigma} = \frac{-\pi}{4}\ e^{-i\sigma}$$

$$I_2 = \frac{-1}{2i}\int_{-\infty}^{\infty} \frac{xe^{-ix}}{x^2-\sigma^2}dx + \frac{\pi}{4}(e^{i\sigma} - e^{-i\sigma}) = \frac{\pi}{2}\ e^{i\sigma}$$

$$\Rightarrow \frac{-1}{2i}\int_{-\infty}^{\infty} \frac{xe^{-ix}}{x^2-\sigma^2}dx = \frac{\pi}{2}\ e^{i\sigma} - \frac{1}{4}(e^{i\sigma} - e^{-i\sigma})$$

$$I(\sigma) = \int_{-\infty}^{\infty} \frac{x \sin x}{x^2-\sigma^2}dx = \frac{1}{2i}\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2-\sigma^2}dx - \frac{1}{2i}\int_{-\infty}^{\infty} \frac{xe^{-ix}}{x^2-\sigma^2}dx$$

$$= \frac{\pi}{2}\ e^{i\sigma} - \frac{\pi}{4}(e^{i\sigma} - e^{-i\sigma}) + \frac{\pi}{2}\ e^{i\sigma} - \frac{\pi}{4}(e^{i\sigma} - e^{-i\sigma})$$

$$= \pi e^{i\sigma} - \frac{\pi}{2}\ e^{i\sigma} + \frac{\pi}{2}\ e^{-i\sigma} = \frac{\pi}{2}(e^{i\sigma} + e^{-i\sigma}) = \pi \cos \sigma$$