Th. A set of vectors containing the null vector $\theta$ in a vector space $V_F$ is linearly dependent.

Proof. Let $S = \{\theta\}$.
Then $S$ is linearly dependent for $c \in F$
$c\theta = \theta$ holds for non-zero $c$.
Let $P$ be an arbitrary set containing the null vector $\theta$.
Suppose $P = \{\alpha_1, \alpha_2 \cdots \alpha_n, \theta\}$. also
Then $c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n + t\theta = \theta$, holds
for $c_i = 0, \forall i$ & $t \neq 0$.
Thus at least one non-zero $t$ exist such that the above relation holds.
Hence $P$ is linearly dependent.

Th. The set consisting of a single non-zero vector $\alpha$ in a vector space $V$ over a field $F$ is linearly independent.

Sol$^n$. Let $S = \{\alpha\}$, $\alpha$ is non-zero vector.
Then $c\alpha = \theta$
$\Rightarrow c = 0$ ~~[illegible]~~
Hence $S$ is linearly independent.

Ex. Determine $k$ so that $S = \{(k,1,1), (1,k,1), (1,1,k)\}$ is linearly independent in $R^3$.

Sol$^n$. Let $c_1(k,1,1) + c_2(1,k,1) + c_3(1,1,k) = \theta = (0,0,0)$
i.e, $(kc_1 + c_2 + c_3, c_1 + c_2 k + c_3, c_1 + c_2 + kc_3) = (0,0,0)$
The set $S$ will be l.ind if $c_1 = 0 = c_2 = c_3$. i.e, the sol$^n$ of
$$\left.\begin{array}{l} kc_1 + c_2 + c_3 = 0 \\ c_1 + c_2 k + c_3 = 0 \\ c_1 + c_2 + kc_3 = 0 \end{array}\right\} \text{ is zero sol}^n.$$

∴ Condition for that is $\begin{vmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{vmatrix} \neq 0$.

i.e. $\begin{vmatrix} k-1 & 1-k & 0 \\ 1 & k & 1 \\ 0 & 1-k & k-1 \end{vmatrix} \neq 0$.

i.e. $(k-1)\{k(k-1)-1(1-k)\}$
$-(1-k)\{1(k-1)-0\} \neq 0$

i.e. $(k-1)[k^2-\cancel{k}-1+\cancel{k}+k-1] \neq 0$

i.e. $(k-1)(k^2+k-2) \neq 0$.

i.e. $(k-1)(k^2+2k-k-2) \neq 0$

i.e. $(k-1)^2(k+2) \neq 0$.

i.e. $k \neq 1, -2$.

Ex.
Show that $S = \{(2,3,1), (2,1,3), (1,1,1)\}$ is linearly dependent in $\mathbb{R}^3$.

Solⁿ. Let $\alpha = (2,3,1)$, $\beta = (2,1,3)$ & $\gamma = (1,1,1)$

Let us consider the relation
$$c_1\alpha + c_2\beta + c_3\gamma = \theta$$

Then $c_1(2,3,1) + c_2(2,1,3) + c_3(1,1,1) = (0,0,0)$

$$\therefore \begin{aligned} 2c_1 + 2c_2 + c_3 &= 0 \\ 3c_1 + c_2 + c_3 &= 0. \\ c_1 + 3c_2 + c_3 &= 0 \end{aligned}$$

The coefficient determinant of the above homoⁿ. system of eqⁿ is

$$A = \begin{vmatrix} 2 & 2 & 1 \\ 3 & 1 & 1 \\ 1 & 3 & 1 \end{vmatrix} \xrightarrow[R_3' = R_3 - R_2]{R_1' = R_1 - R_2} \begin{vmatrix} -1 & 1 & 0 \\ 3 & 1 & 1 \\ -2 & 2 & 0 \end{vmatrix}$$

$= -1(-2+2)$

$= 0$

Hence the above system possesses ~~not~~ non-zero solⁿ. Hence $\exists$ $c_i$'s $(i=1,2,3)$ not all zero s.t $c_1\alpha + c_2\beta + c_3\gamma = \theta$ holds. Thus $S$ is linearly dependent.