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Derivation of Laplace Eqn

In this section, we shall derive one of the most well-known equations in the theory of partial differential equation, the Laplace equation. we derive it from Gravitational potential concept.

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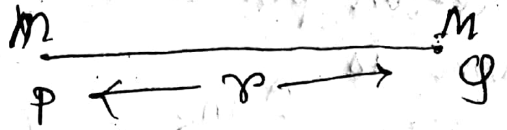
we consider two ~~partial~~ partical of masses m and M, at P and Q as shown in the figure. Let r be the distance between them.



Then, according to Newton's law of gravitation, a force proportional to the product of their masses, and inversely proportional to the square of the distance between them, is given in the form

$$F = G\frac{mM}{r^2}$$ where G is the gravitational constant.

It is customary in potential theory to chose the unit of force so that $G=1$. Thus F becomes $F = \frac{mM}{r^2}$

If $\vec{r}$ represent the vector PQ, the force per unit mass at Q due to the mass at P may be written as

$$\vec{F} = \frac{-m\vec{r}}{r^3} = \nabla\left(\frac{m}{r}\right).$$

which is called the intensity of the gravitational field of force.

we suppose that a partical of unit mass moves under the attraction of the partical of mass m at P from infinity up to Q. The work done by the force $\vec{F}$ is

$$\int_{\infty}^{r} \vec{F}\,d\vec{r} = \int_{\infty}^{r} \nabla\left(\frac{m}{r}\right) d\vec{r} = \frac{m}{r}.$$

This is called the potential at Q due to the partical at P. We denote this by

$$V = -\frac{m}{r}.$$

So that the intensity of force at P is

$$\vec{F} = \nabla\left(\frac{m}{r}\right) = -\nabla V.$$

We shall now consider a number of masses $m_1, m_2, \dots, m_n$, whose distances from Q are $r_1, r_2, \dots, r_n$, respectively. Then the force of attraction per unit mass at Q due to the system is

$$F = \sum_{k=1}^{n} \frac{m_k}{r_k} = \nabla \sum_{k=1}^{n} \frac{m_k}{r_k}.$$

The work done by the forces acting on a partical of unit mass is

$$\int_{\infty}^{r} \vec{F} \cdot d\vec{r} = \sum_{k=1}^{n} \frac{m_k}{r_k} = -V$$

Then the potential satisfies the eqn

$$\nabla^2 V = -\nabla^2 \sum_{k=1}^{n} \frac{m_k}{r_k} = -\sum_{k=1}^{n} \nabla^2\left(\frac{m_k}{r_k}\right) = 0 \quad r_k \neq 0$$

$$\therefore \nabla^2 V = 0.$$

This eqn is called the Laplace equation.

It appears in many physical problem, such as those of electro static potentials, potential in hydrodynamics.