Prob: Canonical form Date - 18.09.20

(1) Consider the eqn

$$4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2$$

Find the canonical form of the given PDE.

⇒ Now comparing the given eqn with

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

where A, B, C, D, E, F and G are function of x, y or constant.

Then we get: $A = 4$, $B = 5$, $C = 1$, $D = 1$, $E = 1$, $F = 0$, $G = 2$

Now $B^2 - 4AC = 5^2 - 4\cdot4\cdot1 = 25 - 16 = 9 > 0$

Since $B^2 - 4AC > 0$ so the given 2nd-order PDE is hyperbolic.

$\xi = \xi(x, y)$
$\eta = \eta(x, y)$
$A^* = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2$
$C^* = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2$

$\therefore A\left(\frac{dy}{dx}\right)^2$

$$A\lambda^2 + B\lambda + C = 0$$

where $\lambda = \left(-\frac{dy}{dx}\right)$.

$\therefore 4\lambda^2 + 5\lambda + C = 0.$

$$\therefore \lambda = \frac{-5 \pm \sqrt{9}}{2\cdot4} = \frac{-5 \pm 3}{8}$$

$\therefore \lambda_1 = -1 \quad \lambda_2 = -\frac{1}{4}$

Then the characteristic eqn

$$\frac{dy}{dx} = -\lambda_1 \quad \Big| \quad \frac{dy}{dx} = -\lambda_2$$

$\therefore \frac{dy}{dx} = 1 \qquad \therefore \frac{dy}{dx} = \frac{1}{4}$

$\therefore y - x = c_1 \qquad \therefore y - \frac{x}{4} = c_2$ where $c_1$ & $c_2$ are arbitrary constant

$\therefore$ Let $\boxed{\xi = y - x}$

$\boxed{\eta = y - \frac{x}{4}}$

$$\xi = y - x$$
$$\eta = y - \frac{1}{4}x$$

$$u_x = \frac{\partial u(\xi,\eta)}{\partial x}$$
$$= \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}.$$
$$= u_\xi(-1) + u_\eta\left(-\frac{1}{4}\right).$$

$$u_{xx} = \frac{\partial}{\partial x}\left(-u_\xi - \frac{1}{4}u_\eta\right).$$
$$= -\frac{\partial u_\xi}{\partial x} - \frac{1}{4}\frac{\partial u_\eta}{\partial x}.$$
$$= -\left(\frac{\partial u_\xi}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u_\xi}{\partial \eta}\frac{\partial \eta}{\partial x}\right) - \frac{1}{4}\left(\frac{\partial u_\eta}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u_\eta}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}\right).$$
$$= -\left\{u_{\xi\xi}(-1) + u_{\xi\eta}\left(-\frac{1}{4}\right)\right\} - \frac{1}{4}\left\{u_{\eta\xi}(-1) + u_{\eta\eta}\left(-\frac{1}{4}\right)\right\}$$
$$= u_{\xi\xi} + \frac{1}{4}u_{\xi\eta} + \frac{1}{4}u_{\eta\xi} + \frac{1}{16}u_{\eta\eta}$$
$$= u_{\xi\xi} + \frac{1}{2}u_{\xi\eta} + \frac{1}{16}u_{\eta\eta}$$

$$u_y = \frac{\partial u(\xi,\eta)}{\partial y}$$
$$= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}$$
$$= u_\xi\cdot 1 + u_\eta\cdot 1$$

$$u_{yx} = \frac{\partial u_\xi}{\partial x} + \frac{\partial u_\eta}{\partial x}$$
$$= \frac{\partial u_\xi}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u_\xi}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} + \frac{\partial u_\eta}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial u_\eta}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}$$
$$= u_{\xi\xi}(-1) + \left(-\frac{1}{4}\right)u_{\xi\eta} + u_{\eta\xi}(-1) + \left(-\frac{1}{4}\right)u_{\eta\eta}$$
$$= -u_{\xi\xi} - \frac{1}{4}u_{\xi\eta} - u_{\eta\xi} + \frac{1}{4}u_{\eta\eta}.$$

$$u_{yy} = \frac{\partial(u_\xi + u_\eta)}{\partial y}$$
$$= \frac{\partial u_\xi}{\partial y} + \frac{\partial u_\eta}{\partial y}$$
$$= \frac{\partial u_\xi}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u_\xi}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial u_\eta}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u_\eta}{\partial \eta}\frac{\partial \eta}{\partial y}$$
$$= u_{\xi\xi} + u_{\xi\eta} + u_{\eta\xi} + u_{\eta\eta}$$

$$4\left(u_{\xi\xi}+\tfrac{1}{2}u_{\xi\eta}+u_{\eta\eta}\right)+5\left(-u_{\xi\xi}-\tfrac{1}{4}u_{\xi\eta}-u_{\xi\eta}-\tfrac{1}{4}u_{\eta\eta}\right)$$
$$+u_{\xi\xi}+2u_{\xi\eta}+u_{\eta\eta}+u_{\xi}-\tfrac{1}{4}u_{\eta}+u_{\xi}+u_{\eta}=2$$

$$\Rightarrow \cancel{4u_{\xi\xi}}+2u_{\xi\eta}+\cancel{\tfrac{4}{4}u_{\eta\eta}}-\cancel{5u_{\xi\xi}}-\tfrac{5}{4}u_{\xi\eta}-5u_{\xi\eta}-\cancel{\tfrac{5}{4}u_{\eta\eta}}$$
$$+\cancel{u_{\xi\xi}}+2u_{\xi\eta}+\cancel{u_{\eta\eta}}-\cancel{u_{\xi}}-\tfrac{1}{4}u_{\eta}+\cancel{u_{\xi}}+u_{\eta}=2$$

$$\Rightarrow -u_{\xi\eta}-\tfrac{5}{4}u_{\xi\eta}=-\tfrac{3}{4}u_{\eta}+2$$

$$\Rightarrow -\tfrac{9}{4}u_{\xi\eta}=-\tfrac{3}{4}u_{\eta}+2$$

$$\Rightarrow \boxed{u_{\xi\eta}=\tfrac{1}{3}u_{\eta}-\tfrac{8}{9}.}$$

(11) Find the canonical form of the 2nd order PDE $u_{xx}-4u_{xy}+4u_{yy}=e^{y}$.

$\therefore$ characteristic eqn

$$\frac{dy}{dx}=-2$$

$\therefore y=-2x+c_1$

$\Rightarrow y+2x=c_1$

$\Rightarrow \boxed{\xi=y+2x}$ $\boxed{\eta=y}$

$$J(\xi,\eta)=\begin{vmatrix}\xi_x & \xi_y\\ \eta_x & \eta_y\end{vmatrix}=\begin{vmatrix}2 & 1\\ 0 & 1\end{vmatrix}\neq 0.$$

$\lambda=2$

$\frac{dy}{dx}=-\lambda$

$\frac{dy}{dx}=-2$

$\Rightarrow y=-2x+c_1$

$\Rightarrow \boxed{y+2x=c}$

$\boxed{\text{Ans}}$ Canonical form

$$u_{\eta\eta}=\frac{1}{4}e^{\eta}.$$

Prob: (3) V.V

Concider the wave eqn

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0 \quad c \text{ is constat}$$

$\therefore u_{tt} - c^2 u_{xx} = 0.$

find the canonical from of wave eqn.

$\Rightarrow$ comparing the given eqn (u is fn $u = u(x, t)$)

$$Au_{xx} + Bu_{xt} + Cu_{tt} + Du_x + Eu_t + Fu = G.$$

Then $A = -c^2 \quad B = 0 \quad C = 1$

$D = 0 \quad E = 0 \quad F = 0 \quad G = 0$

Then $B^2 - 4AC = 0^2 - 4\cdot(-c^2)\cdot 1$

$= 4c^2 > 0$

The wave eqn is hyperbolic everywhere.

The characteristic is

$A\lambda^2 + B\lambda + C = 0$

$\Rightarrow (-c^2)\lambda^2 + 1 = 0$

$\Rightarrow \lambda = \pm\frac{1}{c}.$

$\therefore$ characteristic eqn's are.

$\frac{dt}{dx} = -\lambda_1 \quad \Big| \quad \frac{dt}{dx} = -\lambda_2$

$\therefore \frac{dt}{dx} = -\frac{1}{c}, \quad \frac{dt}{dx} = \frac{1}{c}.$

$\therefore c\,dt = -dx \quad \Big| \quad \therefore c\,dt = dx.$

$\therefore ct + x = c_1. \quad \Big| \quad \therefore ct = x + c_2$

$\therefore \xi = x + ct \quad \quad \therefore x - ct = c_2$

$\therefore \eta = x - ct.$

$$\therefore \boxed{\begin{aligned}\xi &= x + ct \\ \eta &= x - ct\end{aligned}}$$

HW

$$\boxed{u_{\xi\eta} = 0}$$

Bk 18.09.20