Date: 26.08.20

Iteration methods Based on First Degree Equation:

We have already seen that if $f(x)=0$ is a first degree equation in x then it can be readily solved. We now study the iteration methods which will produce exact results whenever $f(x)=0$ in a first degree eqn. Thus, if we approximate $f(x)$ by a first degree equation in the neighbourhood of the root, then we may write—

$$f(x) = a_0 x + a_1 = 0 \;\; (a_0 \neq 0) \rightarrow (1)$$

The Soln of (1) is given by

$$x = -\frac{a_1}{a_0} \rightarrow (2)$$

where $a_0 \neq 0$ and a_1 are arbitrary parameters to be determined by prescribing two appropriate conditions on $f(x)$ and/or its derivatives.

Secant and Regula-Falsi methods:

If x_{k-1} and x_k are two approximations to the root, the we determine a_0 and a_1 in (1) by using the conditions—

- Let $f(x_{k-1}) = f_{k-1}$ and $f(x_k) = f_k$

Now $$f_{k-1} = a_0 x_{k-1} + a_1$$

$$f_k = a_0 x_k + a_1$$

Then $$f_{k-1} - f_k = a_0 (x_{k-1} - x_k)$$

$$\therefore a_0 = \frac{f_{k-1} - f_k}{x_{k-1} - x_k} = \frac{f_k - f_{k-1}}{x_k - x_{k-1}}$$

$$a_1 = \frac{x_k f_{k-1} - x_{k-1} f_k}{x_k - x_{k-1}}$$

Then $x_{k+1} = -\frac{a_1}{a_0}$

$$= -\frac{x_k f_{k-1} + x_{k-1} f_k}{(x_k - x_{k-1})} \times \frac{x_k - x_{k-1}}{f_k - f_{k-1}}$$

$$= \frac{x_{k-1} f_k - x_k f_{k-1}}{f_k - f_{k-1}}$$

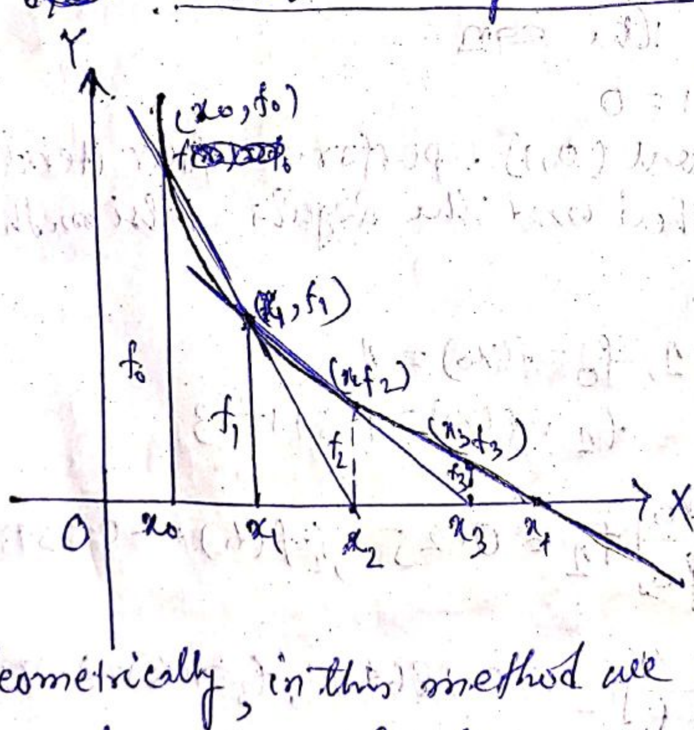
$$\therefore x_{k+1} = \frac{x_{k-1} f_k - x_k f_k + x_k f_k - x_k f_{k-1}}{f_k - f_{k-1}}$$

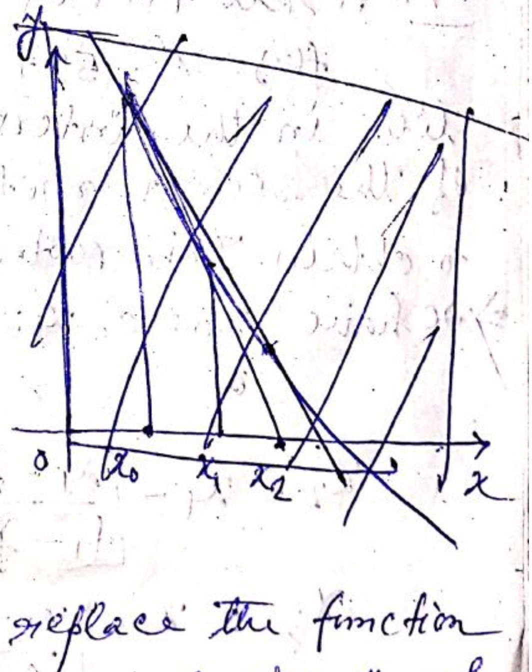
$$= \frac{f_k (x_{k-1} - x_k) + x_k (f_k - f_{k-1})}{(f_k - f_{k-1})}$$

$$= x_k - \frac{(x_k - x_{k-1})}{(f_k - f_{k-1})} f_k, \quad k = 1, 2 \ldots$$

This is called the secant or the chord method.

VII Geometrically Discuss:

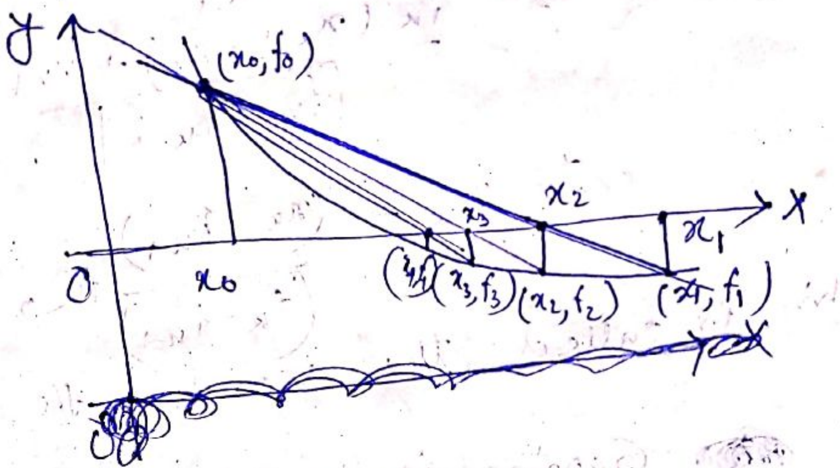




Geometrically, in this method we replace the function $f(x)$ by a straight line or a chord passing through the point (x_k, f_k) and (x_{k-1}, f_{k-1}) and take the point of intersection of the straight line with the x-axis as the next approximation to the root.

Regula - Falsi method

If the approximations are such that $f_k f_{k-1} < 0$, then $\boxed{x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f_k - f_{k-1}}}$ is known as the Regula - Falsi method. Since (x_{k-1}, f_{k-1}), (x_k, f_k) are known before the start of the iteration, the secant and the Regular-falsi method require one function per iteration.



Ex: A real root of the eqn

$f(x) = x^3 - 5x + 1 = 0$

lies in the interval $(0, 1)$. perform four iterations of the secant method and the Regula - Falsi method to obtain this root.

$\Rightarrow$ we have, $x_0 = 0$, $x_1 = 1$, $f_0 = f(x_0) = 1$

$f_1 = f(x_1) = 1 - 5 + 1 = -3$

$$x_2 = x_1 - \left[\frac{x_1 - x_0}{f_1 - f_0}\right] f_1 = 0.25, \; f_2 = f(x_2) = -0.234375$$

$$x_3 = x_2 - \left[\frac{x_2 - x_1}{f_2 - f_1}\right] f_2 = 0.186491, \; f_3 = f(x_3) = 0.074276$$

$$x_4 = x_3 - \left[\frac{x_3 - x_2}{f_4 - f_2}\right] f_3 = 0.201736, \; f_4 = f(x_4) = -0.000470$$

$$x_5 = x_4 - \left[\frac{x_4 - x_3}{f_1 - f_3}\right] f_1 = 0.201640$$

Regula-Falsi method:

$x_2 = x_1 - \left[\frac{x_1 - x_0}{f_1 - f_0}\right] f_1 = 0.25 \quad f_2 = f(x_2) = -0.234375$

Now $f(x_0)\, f(x_2) < 0 = 1\, f(0.25) = 1 \times (-0.234375) < 0$

$\therefore \xi \in (x_0, x_2)$

$\therefore x_3 = x_2 - \frac{[x_2 - x_0]}{(f_2 - f_0)} f_2 = 0.202532, \; f_3 = f(x_3) = -0.004352$

Since $f(x_0) f(x_3) < 0$, $\xi \in (x_0, x_3)$.

$\therefore x_4 = x_3 - \left[\frac{x_3 - x_0}{f_3 - f_0}\right] f_3 = 0.201659, \; f_4 = f(x_4) = -0.000070$

Since $f(x_0) f(x_4) < 0$, $\xi \in (x_0, x_4)$. Therefore

$x_5 = x_4 - \left[\frac{x_4 - x_0}{f_4 - f_0}\right] f_4 = 0.201640$

III Newton - Raphson method:

To find the soln of $f(x)=0$, we approximate $f(x)$ to be a linear eqn $f(x) = a_0 x + a_1$

Let x_k be a soln of $f(x)=0$ then

$$f_k = f(x_k) = a_0 x_k + a_1 \rightarrow (1)$$

Now $f'_k = a_0 \rightarrow (2)$

$\therefore x_k f'_k = a_0 x_k$

$\therefore$ from (1) $\quad f_k = x_k f'_k + a_1$

$\therefore a_1 = f_k - x_k f'_k$

Again if x_{k+1} is the next approximation of a root then $f(x_{k+1}) = 0$

$$\Rightarrow a_0 x_{k+1} + a_1 = 0$$

$$\Rightarrow x_{k+1} = -\frac{a_1}{a_0} = -\frac{f_k - x_k f'_k}{f'_k}$$

$$\Rightarrow x_{k+1} = \frac{x_k f'_k}{f'_k} - \frac{f_k}{f'_k}$$

$$\Rightarrow \boxed{x_{k+1} = x_k - \frac{f_k}{f'_k}} \quad k = 0, 1, \ldots$$

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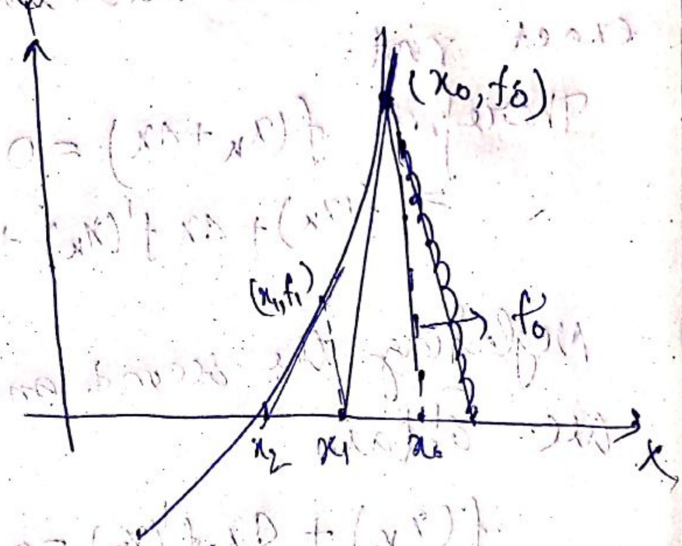
Geometrical Significant:

Let x_k be the initial approximation and $f_k = f(x_k)$ be the corresponding function value. Then (x_k, f_k) be any point on the curve. $f'_k = a_0$ is the slope of the curve at (x_k, f_k). At it intersect with x-axis x_{k+1} then x_{k+1} is the next approximate root.



This method is called The Newton-Raphson method. The method may also be obtained directly from $x_{k+1} = x_k - \left[\frac{x_k - x_{k-1}}{f_k - f_{k-1}}\right]$

by taking the limit $x_{k-1} \to x_k$. In the limite when $x_{k-1} \to x_k$, then the cord passing through the points (x_k, f_k) and (f_{x-1}, f_{k-1}) become the tangent at the point (x_k, f_k). Thus, in this case the problem of finding the root of the eqn $f(x) = 0$ is equivalent to finding the point of intersection of the tangent to the curve $y=f(x)$ at the point (x_k, f_k) with the x-axis. The Newton - Raphson method requires two evaluations f_k, f'_k for each iteration.

Alternative :

Let x_k be an approximation to the root of the equation $f(x)=0$. Let Δx be an increment in x such that $x_k+\Delta x$ is an exact root.

Therefore $f(x_k+\Delta x)=0$

$$\Rightarrow f(x_k)+\Delta x\, f'(x_k)+\frac{1}{2!}(\Delta x)^2 f''(x_k)+\cdots=0$$

(Taylor series expantion)

Neglating the second and higher powers of Δx, we obtain

$$f(x_k)+\Delta x\, f'(x_k)=0$$

$$\Delta x \simeq -\frac{f(x_k)}{f'(x_k)}$$

$\therefore$ Hence $x_{k+1} = x_k+\Delta x = x_1 - \dfrac{f(x_k)}{f'(x_k)}$

$$\therefore \boxed{x_{k+1} = x_1 - \frac{f(x_k)}{f'(x_k)}}$$