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VII Some problem

① perform four iterations of the Newton-Raphson method to find the smallest possitive root of the equation $f(x) = x^3 - 5x + 1 = 0$. The smallest possitive root lies in the interval $(0,1)$.

Soln Take the initial approximation as $x_0 = 0.5$
we have
$$f(x) = x^3 - 5x + 1$$
$$f'(x) = 3x^2 - 5$$
using the Newton-Raphson method:
$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$
$$= x_k - \frac{x_k^3 - 5x_k + 1}{3x_k^2 - 5} = \frac{2x_k^3 - 1}{3x_k^2 - 5}, \quad k = 0, 1, \ldots$$
Starting with $x_0 = 0.5$, we obtain
$$x_1 = 0.176471, \quad x_2 = 0.201568$$
$$x_3 = 0.201640, \quad x_4 = 0.201640$$
the exact value correct to six decimal places is 0.201640.

② perform four iterations of the Newton-Raphson method to obtain the approximate value of $(17)^{1/3}$ starting with the initial approximation $x_0 = 2$.

Soln: Let $x = (17)^{1/3}$ we obtain $x^3 = 17$
Let $f(x) = x^3 - 17 = 0$. using Newton-Raphson method
$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, \ldots$$
we get
$$x_{k+1} = x_k - \frac{x_k^3 - 17}{3x_k^2} = \frac{2x_k^3 + 17}{3x_k^2}$$
Starting with $x_0 = 2$, we obtain
$$x_1 = \frac{2x_0^3 + 17}{3x_0^2} = 2.75, \quad x_2 = 2.582646, \ldots$$
$$x_4 = 2.571282$$

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(3) Apply N-R'S method to determine a root of the equation $f(x) = \cos x - xe^{x} = 0$ such that $|f(x^*)| < 10^{-8}$ where x^* is the approximation to the root. Take the initial approximation as $x_0 = 1$ (home work)

(4) Show that the initial approximation x_0 for finding $1/N$, where $N \in \mathbb{Z}$ by the Newton-Raphson method must satisfy $0 < x_0 < \frac{2}{N}$, for convergence.

Sol^n: we write $f(x) = \frac{1}{x} - N = 0$.

Then Newton-Raphon method become —

$$f'(x) = -\frac{1}{x^2}$$

$$\therefore x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$= x_k - \frac{\frac{1}{x_k} - N}{-\frac{1}{x_k^2}}$$

$$= x_k + \frac{(1 - Nx_k)}{x_k} \times x_k^2$$

$$= x_k + x_k - Nx_k^2$$

$$\therefore \boxed{x_{k+1} = 2x_k - Nx_k^2}$$

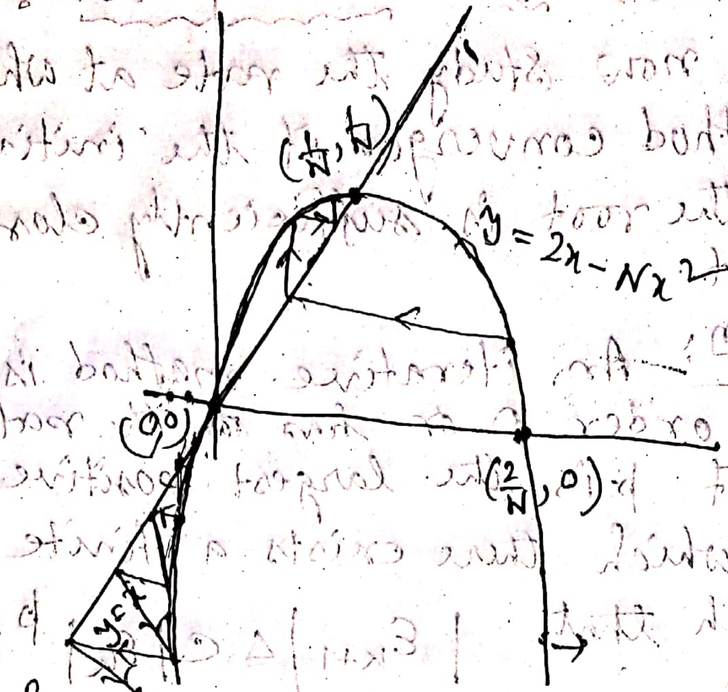
Let us now draw the graph of $y = x$, $y = 2x - Nx^2$. Then second eq^n is the parapala —

$$Nx^2 - 2x = -y.$$

$$\therefore x^2 - \frac{2}{N}x = -\frac{y}{N}$$

$$\therefore x^2 - 2\cdot x\cdot\frac{1}{N} + \frac{1}{N^2} = -\frac{y}{N} + \frac{1}{N^2}$$

$$\therefore \left(x - \frac{1}{N}\right)^2 = -\frac{1}{N}\left(y - \frac{1}{N}\right)$$



From the graph, the point of intersection of these two curve $y = x$ and $y = 2x - Nx^2$ is the required value of $1/N$. So we chose initial approximation such a waya that tangent must converges to the point $(\frac{1}{N}, \frac{1}{N})$.

Now we find that for any initial approximation out side the range $0 < x_0 < 2/N$, the method diverges. If $x = 0$, the iterations does not converges to $1/N$ but remain zero always. This shows the importance of choosing a suitable initial approximation.

Rate of convergence:

We now study the rate at which the iteration method converges if the initial approximation to the root is sufficiently close to the desired root.

Defn: An iterative method is said to be of order p or has the rate of convergence p if p is the largest positive real number for which there exists a finite constant $c \neq 0$ such that $|\varepsilon_{k+1}| \le c |\varepsilon_k|^p$.

where $\varepsilon_k = x_k - \xi$ is the error in the k-th iterate.

The constant c is called the asymptotic error constant and usually depends on derivatives of $f(x)$ at $x = \xi$.

Rate of convergence of N-R method

Let $f(x) = 0$ be a eqn and ξ is the correct root and x_k be the k-th approximate root. Then $x_k = \xi + \varepsilon_k$ where ε_k is the error in k-th approximation.

Then $f(x_k) = f(\xi + \varepsilon_k)$ and

Now $x_{k+1} = x_k - \dfrac{f(x_k)}{f'(x_k)}$

$$\Rightarrow \xi + \varepsilon_{k+1} = \xi + \varepsilon_k - \frac{f(\xi + \varepsilon_k)}{f'(\xi + \varepsilon_k)}$$

$$\Rightarrow \varepsilon_{k+1} = \varepsilon_k - \frac{f(\xi) + \varepsilon_k f'(\xi) + \frac{1}{2!} f''(\xi)\varepsilon_k^2 + \cdots}{f'(\xi) + \varepsilon_k f''(\xi) + \frac{1}{2!}\varepsilon_k f'''(\xi) + \cdots}$$

$$= \varepsilon_k - \frac{\varepsilon_k f'(\xi) + \frac{1}{2} f''(\xi)\varepsilon_k^2 + \cdots}{f'(\xi) + \varepsilon_k f''(\xi) + \cdots}$$

$$= \varepsilon_k - \left[\varepsilon_k + \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}\varepsilon_k^2 + \cdots\right]\left[1 + \frac{f''(\xi)}{f'(\xi)}\varepsilon_k + \cdots\right]^{-1}$$

$$= \varepsilon_k - \left[\varepsilon_k + \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}\varepsilon_k^2 + \cdots\right]\left[1 - \frac{f''(\xi)}{f'(\xi)}\varepsilon_k \cdots\right].$$

$$= \varepsilon_k - \left[\varepsilon_k + \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}\varepsilon_k^2 - \frac{f''(\xi)}{f'(\xi)}\varepsilon_k^2\right]$$

$$= \cancel{\varepsilon_k} - \cancel{\varepsilon_k} + \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}\varepsilon_k^2 + O(\varepsilon_k^3).$$

(on neglecting ε_k^3 and higher power of ε_k).

$$\varepsilon_{k+1} = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}\varepsilon_k^2$$

$$\therefore \varepsilon_{k+1} = C\,\varepsilon_k^2$$

where $C = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}$

Thus, the newton-Raphson method has second order convergence.