02.09.20 General Iteration methods

Theorem: If $\varphi(x)$ is a continuous function in some interval $[a,b]$ that contains the root and $|\varphi'(x)| \le c < 1$ in the interval, then for any choice of $x_0 \in [a,b]$, the sequence $\{x_k\}$ determined from $x_{k+1} = \varphi(x_k)$, $k = 0,1,2,\ldots$ converges to the root ξ of $x = \varphi(x)$.

Proof: The exact solution ξ satisfies the equation

$$\xi = \varphi(\xi).$$

we have $\xi - x_{k+1} = \varphi(\xi) - \varphi(x_k)$, $k = 0,1,2\ldots$.

Using the mean value theorem, we get

$$\xi - x_{k+1} = (\xi - x_k)\varphi'(\xi_k); \quad x_k < \xi_k < \xi.$$

$$\Rightarrow \varepsilon_{k+1} = \varepsilon_k \varphi'(\xi_k)$$

$$= \varepsilon_{k-1}\varphi'(\xi_k)\varphi'(\xi_{k-1}).$$

$$= \varepsilon_0 \varphi'(\xi_k)\varphi'(\xi_{k-1})\ldots\varphi'(\xi_0).$$

where $x_0 < \xi_0 < \xi$, $x_1 < \xi_1 < \xi \ldots x_{k-1} < \xi_{k-1} < \xi$.

if $|\varphi'(\xi_r)| \le c$, $r = 0,1,2,\ldots k-1$

then $|\varepsilon_{k+1}| \le |\varepsilon_0| c^{k+1}$

If $C<1$, the right hand side goes to zero as k become large. Then, the iteration method converges if $|\varphi'(x)| \le C < 1$. This condition is same as the Lipschitz condition and C is the Lipschitz constant.

VII Second order Method:

Here from the eq^n

$$\varepsilon_{k+1} = \varepsilon_k \varphi'(\xi_k) + \frac{\varepsilon_k^2}{2!}\varphi''(\xi)$$

if $\varphi'(\xi) = 0$.

Then $\varepsilon_{k+1} = \frac{\varepsilon_k^2}{2!}\varphi''(\xi)$.

$$\varepsilon_k = \frac{\varepsilon_{k-1}^2}{2!}\varphi''(\xi)$$

$$\varepsilon_{k-1} = \frac{\varepsilon_{k-2}^2}{2!}\varphi''(\xi)$$

$$\vdots$$

$$\varepsilon_1 = \frac{\varepsilon_0^2}{2!}\varphi''(\xi)$$

$$\therefore \varepsilon_{k+1} = \frac{\varphi''(\xi)}{2!} \times \frac{\varepsilon_{k-1}^{2^2}}{2!^2}\{\varphi''(\xi)\}^2$$

$$= \frac{\varepsilon_{k-1}^{2^2}}{2!^{2^2-1}}(\varphi''(\xi))^{2^2-1}$$

$$= \left\{\frac{\varphi''(\xi)}{2!}\right\}^{2^k-1} \varepsilon_0^{2^k}$$

if $\left|\frac{\varphi''(\xi)}{2!}\right| \le 1$, ε_0 is small, then the iteration method converges and has second order convergence, each successive iteration approximately doubles the number of significant digits of accuracy.

III Def^n of order of iteration).

The iteration method $x_{k+1} = \varphi(x_k)$ is said to be of the p-th order if

$$\varphi'(\xi) = \varphi''(\xi) = \cdots = \varphi^{(p-1)}(\xi) = 0$$

$$\varphi^{p}(\xi) \neq 0$$

where ξ is the solution of $x = \varphi(x)$.

<u>Note:</u> The eq^n ~~$x_{k+1} = \varphi(x_k)$ for a p-th~~

$$\varepsilon_{k+1} = \frac{1}{p!}\varphi^{p}(\xi)\,\varepsilon_k^{\,p} + O(\varepsilon_k^{\,p+1})$$

is ~~the~~ error eq^n for p-th order iteration method.

Thus the number of significant digits of accuracy at each step is approximately p-times the number of significant digits of accuracy of the previous step.

<u>prob:</u> Show that Newton-Raphson method is second order iteration method.

$\Rightarrow$ The iteration formula for Newton-Raphson method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = \varphi(x_k).$$

we have

$$\varphi(x) = x - \frac{f(x)}{f'(x)}$$

$$\varphi'(x) = 1 - \frac{[f'(x)]^2 - f(x)f''(x)}{[f'(x)]^2} = \frac{f(x)\,f''(x)}{[f'(x)]^2}$$

$$\varphi''(x) = \frac{1}{[f'(x)]^3}\Big[f'(x)\{f'(x)f''(x) + f(x)f'''(x)\} - 2f(x)(f''(x))^2\Big]$$

since ξ is an exact root, we have

$f(\xi) = 0$, $f'(\xi) \neq 0$

we find that

$\varphi(\xi) = \xi$, $\varphi'(\xi) = 0$, and $\varphi''(\xi) = \frac{f''(\xi)}{f'(\xi)} \neq 0$

Therefore, the Newton-Raphson method is a second order method.

<u>Ex:</u> The eqn $f(x) = 3x^3 + 4x^2 + 4x + 1 = 0$ has a root in the interval $(-1, 0)$. Determine an iteration for $\varphi(x)$, such that the seqn of iterations obtained from

$$x_{k+1} = \varphi(x_k),\ x_0 = -0.5,\ k = 0, 1, 2, \ldots$$

converges to the root.

<u>Soln:</u> we write the given equation as

$$x = x + \alpha(3x^3 + 4x^2 + 4x + 1) = \varphi(x)$$

where α is an arbitrary constant to be determined such that

$$|\varphi'(x)| = |1 + \alpha(9x^2 + 8x + 4)| < 1$$

for all $x \in (-1, 0)$.

Sing $g(x) = 9x^2 + 8x + 4 > 0\ \forall\ x \in (-1, 0)$

$; 1 + \alpha g(x) < 1 \Leftrightarrow \alpha g(x) < 0 \Leftrightarrow \alpha < 0.$

$[\because g(x) > 0]$

$\therefore$ The condition $|\varphi'(x)| < 1$ must also be satisfied at the initial approximation $x_0 = -0.5$, using this condition, we get $|\varphi'(-0.5)| = \left|1 + \frac{9\alpha}{4}\right| < 1$

$$\Rightarrow -1 < 1 + \frac{9\alpha}{4} < 1$$

$$\Rightarrow -2 < \frac{9\alpha}{4} < 0$$

$$\Rightarrow -\frac{8}{9} < \alpha < 0$$

$\therefore$ The range of α depend on x_0.

For $\alpha = -\frac{1}{2}$.

$$x_{k+1} = x_k - \frac{1}{2}(3x_k^3 + 4x_k^2 + 4x_k + 1)$$
$$= -\frac{1}{2}(3x_k^3 + 4x_k^2 + 2x_k + 1) = \varphi(x_k).$$

Starting with $x_0 = -0.5$, we get

$x_1 = -0.3125$

$x_2 = -0.337036$

$x_3 = -0.332723$

$x_4 = -0.333435$

$x_5 = -0.333326$

$\therefore$ each step $|\varphi'(x_j)| < 1$

$\therefore$ exact root is $-\frac{1}{3}$.

Year Question:

(1) Show that regula-falsi method converges linearly (Y-2018, 4 marks)

(2) What do you mean by order of convergence of an iterative method? (Y-2018, 2 marks)

(3) If α and β are two real roots of the quadratic equation $ax^2+bx+c=0$ ($a \neq 0$), show that the iteration method

$$x_{k+1} = -\frac{b}{x_k + a}$$ is converges near $x = \alpha$

if $|\alpha| < |\beta|$

(Y-2018, 3 marks)

(4) Deduce the condition of convergence of the fixed point iteration process. Justify the name 'fixed point' (4+1) (2018)