

Date
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System of Linear Algebraic Equation and Eigenvalue Problems

$$\text{Let } 2x + 3y + 5z = 1$$

$$5x + z = 1$$

$$3y + 2z = 2$$

This is called system of linear Algebraic Equation in 3 unknown (x, y, z) and 3 eqn

This eqn can be written as

$$AX = b \quad \text{where } A = \begin{pmatrix} 2 & 3 & 5 \\ 5 & 0 & 1 \\ 0 & 3 & 2 \end{pmatrix}$$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

Consider a system of n linear algebraic eqn in n -unknown.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

where a_{ij} ($i, j = 1, 2, \dots, n$) are the known coefficients
 b_i ($i = 1, 2, \dots, n$) are known values and x_i ($i = 1, 2, \dots, n$)
are the unknowns to be determined.

Then this system of eqn in n unknown and
 n eqn can be written as matrix form

$AX = b$ where $A = (a_{ij})_{n \times n}$ and $b = (b_i)_{1 \times n}$
 and $X = (x_i)_{1 \times n}$
 all can write

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Notation:

$A \rightarrow$ Square matrix of order n .

$a_{ij} \rightarrow$ element in the i th row and j -th column of the matrix A .

$A^{-1} \rightarrow$ inverse of A

$A^T \rightarrow$ transpose of A

$|A| \rightarrow$ determinant of A

$A_{ij} \rightarrow$ cofactor of a_{ij} in A .

$O \rightarrow$ null matrix.

$I \rightarrow$ identity matrix of order n .

$D \rightarrow$ Diagonal matrix of order n .

$L \rightarrow$ lower triangular matrix of order n .

$\|A\| \rightarrow$ norm of A

Defination: Let A be a real matrix. Then

(1) A is said to be - $(A = (a_{ij})_{n \times n})$
 nonsingular: if $|A| \neq 0$. $a_{ij} \in \mathbb{R}$.

(2) symmetric: if $A^T = A$.

(3) skew-symmetric: if $A^T = -A$.


(4) orthogonal: if $AA^T = I$ or $A^T = A^{-1}$.

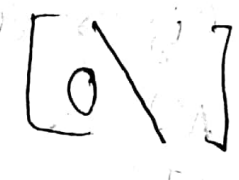
~~(5) null~~

(5) null: if $A = (a_{ij})_{n \times n}$
 $a_{ij} = 0 \quad i, j = 1(1)n$.

(6) diagonal: if $a_{ij} = 0, i \neq j$.

(7) unit matrix: if $a_{ij} = 0 \quad i \neq j \quad a_{ii} = 1 \quad i = 1(1)n$.

(8) lower triangular: if $a_{ij} = 0 \quad j > i$ 

(9) upper triangular: if $a_{ij} = 0 \quad i > j$ 

10. diagonally dominant:

$$\text{if } |a_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^n |a_{ij}| \quad i = 1(1)n.$$

11. Let $A = (a_{ij})_{n \times n}$, $a_{ij} \in \mathbb{C} \quad i, j = 1(1)n$.
 be a complex matrix -
 Then A is called

(1) Hermitian: when $(\overline{A})^T = A$

(2) Skew Hermitian: when $(\overline{A})^T = -A$

(3) unitary if $A^{-1} = (\overline{A})^T = A^*$, $A^* = A^{-1}$

(4) normal if $AA^* = A^*A$.

Linear system of Equation:

In the matrix notation, the previous system can be written as

$$AX = b \rightarrow \textcircled{1}$$

where A is called coefficient matrix.

Then $[A|b]$ is called the augmented matrix.

It is formed by appending the column b to the $n \times n$ matrix A .

If all b_i are zero (i.e., $b=0$), then the system $\textcircled{1}$ is called (i.e., $AX=0$) homogeneous system and if

$b_i \neq 0$ for some $i=1(1)n$ then the system $\textcircled{1}$ is called in-homogeneous or non-homogeneous system of eqn.

The inhomogeneous system $\textcircled{1}$ has a unique solution

iff $|A| \neq 0$.

$$\text{i.e. } \det A = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \neq 0.$$

The solution of the system $\textcircled{1}$ may be written as

$$X = A^{-1}b.$$

If the system is homogeneous and $|A| \neq 0$ then the only soln is $X=0$.

The methods of solution of the linear algebraic eqn $\textcircled{1}$ may be classified into two types.

- (i) Direct method: (Cramer Rule / triangular meth)
- (ii) Iterative method: (Gauss Elimination method)