System of Equation

The methods of Solution of the linear algebric eqn are (1) Direct method: ~~cr~~ Cramer Rul:

(2) Iterative method.

Direct methods

Linear Algebra by S. Friedberg, A. Insel, L. Spence

Let $AX=b$ be a system of eqn with n-unknowns $(x_1, x_2 \ldots x_n)$ and n eqn.

Then the system is called consistent if $\text{rank}(A) = \text{rank}(A|b)$.

The system of eqn $AX=b$ can be solved in the following cases

(i) $A = D$ (diagonal matrix).

$$A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \cdots & & & \cdots \\ 0 & \cdots & & a_{nn} \end{pmatrix}$$

Then System of eqn become.

$$a_{11}x_1 + 0x_2 \ldots + 0x_n = b_1$$
$$0x_1 + a_{22}x_2 + \ldots 0x_n = b_2$$
$$\text{---}$$
$$0x_1 + 0x_2 + \ldots + a_{nn}x_n = b_n.$$

$\therefore$ From first $x_1 = \dfrac{b_1}{a_{11}}$ $\quad (a_{11}, a_{22}, \ldots, a_{nn} \neq 0)$.

$$x_2 = \frac{b_2}{a_{22}}$$
$$\vdots$$
$$x_{nn} = \frac{b_n}{a_{nn}}.$$

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(ii) A = L (Lower triangular).

Let $Ax = b$ be a system of n eqn and n unknown. if A is lower triangular matrix then

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 & \cdots \\ a_{21} & a_{22} & \cdots & 0 & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots & \\ a & \cdots & & & \\ a_{n1} & a_{n2} & \cdots & & a_{nn} \end{bmatrix}$$

Then System become.

$$a_{11}x_1 = b_1 \Rightarrow x_1 = \frac{b_1}{a_{11}}$$

$$a_{22}x_1 + a_{22}x_2 = b_2$$

$$\cdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

Solving the first eqn and then successively solving the second, third and so on, we obtain.

$$x_1 = \frac{b_1}{a_{11}}$$

$$x_2 = \frac{b_2 - a_{21}x_1}{a_{22}} = (b_2 - a_{21}x_1)\frac{1}{a_{22}}$$

$$x_3 = (b_3 - a_{31}x_1 - a_{32}x_2)\frac{1}{a_{33}}$$

$$\vdots$$

$$x_n = \left(b_n - \sum_{j=1}^{n-1} a_{nj}x_j\right)/a_{nn}$$

where $a_{ii} \neq 0$, $i = 1(1)n$.

Since the unknown are solved by forward substitution, this method is called forward substitution method.

(ii) Let $A = U$ (upper triangular).

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ \cdots & & & a_{nn} \end{pmatrix}$$

Cramer Rule:

Let $Ax = b$ be a system of eqn with n eqn and n - unknown.

Then $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ - & - & - & - \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}_{n\times n}$.

In order to determine the unknow x_i from the system of eqn, we multiply the eqn's by the cofactors A_{i1} $i = 1(1)n$. then we get

$$\left(\sum_{i=1}^{n} a_{i1} A_{i1}\right) x_1 + \sum_{j=2}^{n} \left(\sum_{i=1}^{n} a_{ij} A_{i1}\right) x_j = \sum_{i=1}^{n} b_i A_{i1}.$$

which gives

$$|A|\, x_1 = b_1 A_{11} + b_2 A_{21} + \cdots + b_n A_{n1}$$

$$\vdots$$

$$|A|\, x_n = b_1 A_{1n} + b_2 A_{2n} + \cdots + b_n A_{nn}.$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \underline{A}^{-1} b$$

Iterative Method-

(I) Gauss Elimination method:

Here, the unknown are eliminated by combining eqn's such that n eqn in n unknowns are reduced to an equivalent upper triangular system which is then solved by back substitution method.

Consider a system of 3 eqn in 3 unknowns.

$$\left.\begin{array}{l} \boxed{a_{11}}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + \boxed{a_{22}}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + \boxed{a_{33}}x_3 = b_3 \end{array}\right\} \rightarrow (1)$$

In the first stage of elimination, multiply the first row in (1) by $\frac{a_{21}}{a_{11}}$, $\frac{a_{31}}{a_{11}}$ respectively and subtract from the second and third rows. we get.

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{22}^{(2)}x_2 + a_{23}^{(2)}x_3 = b_2^{(2)} \\ a_{32}^{(2)}x_2 + a_{33}^{(2)}x_3 = b_3^{(2)} \end{array}\right\} \rightarrow (2)$$

where $a_{22}^{(2)} = a_{22} - \frac{a_{21}}{a_{11}}$, $a_{23}^{(2)} = a_{23} - \frac{a_{21}}{a_{11}}a_{13}$

$a_{32}^{(2)} = a_{32} - \frac{a_{31}}{a_{11}}$, $a_{33}^{(2)} = a_{33} - \frac{a_{31}}{a_{11}}a_{33}$

$b_2^{(2)} = b_2 - \frac{a_{21}}{a_{11}}b_1$, $b_3^{(2)} = b_3 - \frac{a_{31}}{a_{11}}b_1$

In the second stage of elimination, multiply the first row ~~by~~ second by $\frac{a_{32}^{(2)}}{a_{22}^{(2)}}$ and subtract from 3rth. row we get

~~a_{33}~~

$$a_{11}^{(1)}x_1 + a_{12}^{(1)}x_2 + a_{13}^{(1)}x_3 = b_1$$
$$a_{22}^{(2)}x_2 + a_{23}^{(2)}x_3 = b_2^{(2)}$$
$$a_{33}^{(3)}x_3 = b_3^{(3)}$$

Where
$$a_{33}^{(3)} = a_{33}^{(2)} - \frac{a_{32}^{(2)}}{a_{22}^{(2)}} a_{23}^{(2)}$$
$$b_3^{(3)} = b_3^{(2)} - \frac{a_{32}^{(2)}}{a_{22}^{(2)}} b_2^{(2)}$$

The System (1) is beome a upper triangular system and we can be solved using the back substitution method.

The elimination procedure described above to determine the unknow is called Gauss Elimination method.

Gauss elimination method given.

: The Gauss ellimination method

$$[A|b] \xrightarrow[\text{Ellimination}]{\text{Gauss}} [U|C].$$

The elements $a_{11}^{(1)}, a_{22}^{(2)}, \& a_{33}^{(3)}$ which have been assumed to be non-zero are called $\boxed{\text{pivot elements}}$

If we apply this iteration method for n-eqn in n unknow system

Then the iteration formula.

$$b_i^{(k)} = a_{i,n+1}^{(k)}, \quad i,k = 1(1)n$$

The elements $a_{ij}^{(k)}$, with $i,j \geq k$.

$$a_{ij}^{(k+1)} = a_{ij}^{(k)} - \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}} a_{kj}^{(k)}.$$

$i = k+1, k+2, \ldots n, \quad j = k+1, \ldots n, n+1$