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System of Linear Algebric Eqn

System of Linear Algebraic eqn

consider a system of n linear Algebraic eqn in n unknowns

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}\right\} \rightarrow (1)$$

we can write this eqn as matrix form
$Ax = b$ where $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$.

$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ $b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$

In Gauss-Jordan Elimination method, the coefficient matrix (A). is reduced to a diagonal matrix rather than a triangular matrix. At all steps of the Gauss-elimination method, the elimination is done not only in the equation below but also in the equation above the pivots, producing the soln without using the back substitution method.

On the completion of Gauss-Jordan method the equation's (1) become -

$$\begin{bmatrix} a_{11}^{(k)} & 0 & \cdots & 0 \\ 0 & a_{22}^{(k)} & \cdots & 0 \\ 0 & 0 & a_{33}^{(k)} \cdots & 0 \\ \vdots & & & \vdots \\ & & & a_{nn}^{(k)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d_1^{(k)} \\ d_2^{(k)} \\ \vdots \\ d_n^{k} \end{bmatrix}$$

The solution is given by

$$x_i = \frac{d_i^{(k)}}{a_{ii}^{(k)}} \quad i = 1(1)n.$$

Hence, the Gauss-Jordan method gives

$$[\underline{A} | \underline{b}] \xrightarrow[\text{Jordan}]{\text{Gauss}} [\underline{D} | \underline{d}]$$

① Find the inverse of the coefficient metrix of the system

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

Ans: $A^{-1} = \begin{bmatrix} \frac{7}{5} & \frac{1}{5} & -\frac{2}{5} \\ -3/2 & 0 & 1/2 \\ 11/10 & -1/5 & -1/10 \end{bmatrix}$.

LU-decomposition Method.

consider the equations.

$$\boxed{\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ 4x_1 + 3x_2 - x_3 &= 6 \\ 3x_1 + 5x_2 + 3x_3 &= 4 \end{aligned}}$$

use the decomposition method.

Here the coeff^n matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$$

Now we want to write

$A = LU$ where $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

we choose $u_{ii} = 1$, $i = 1(1)3$

$$LU = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$LU = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12}+l_{22} & l_{21}u_{13}+l_{22}u_{23} \\ l_{31} & l_{31}u_{12}+l_{32} & l_{31}u_{13}+l_{32}u_{23}+l_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$$

on comparing the corresponding elements we obtain—

1st column: $l_{11} = 1 \quad l_{21} = 4 \quad l_{31} = 3$

1st row: $l_{11}u_{12} = 1 \Rightarrow u_{12} = 1.$

$l_{11}u_{13} = 1 \Rightarrow u_{13} = 1.$

2nd column: $l_{21}u_{12} + l_{22} = 3 \Rightarrow l_{21}\cdot 1 + l_{22} = 3$

$\Rightarrow 4 + l_{22} = 3$

$\Rightarrow l_{22} = -1.$

$l_{31}u_{12} + l_{32} = 5$

$\Rightarrow 3 + l_{32} = 5$

$\Rightarrow l_{32} = 2.$

2nd row: $l_{21}u_{13} + l_{22}u_{23} = -1$

$\Rightarrow 4 + (-1)u_{23} = -1$

$\Rightarrow u_{23} = 5.$

$l_{31}u_{13} + l_{32}u_{23} + l_{33} = 3$

$\Rightarrow 3\cdot 1 + 5\cdot 2 + l_{33} = 3$

$\therefore l_{33} = -10.$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ 3 & 2 & -10 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

The given eqn $AX = b$

$\Rightarrow LUX = b$

$\Rightarrow L(UX) = b$

$\Rightarrow \boxed{LZ = b} \quad [\,\underline{UX = Z}\,]$

where $Z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$

$LZ = b$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ 3 & 2 & -10 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

using forward Substitution

$\Rightarrow$ $z_1 = 1$

$4z_1 - z_2 = 6 \Rightarrow z_2 = 4 - 6 = -2$

$3z_1 + 2z_2 - 10z_3 = 4$

$\Rightarrow -10z_3 = 4 - 3 + 4$

$= 5$

$\therefore z_3 = \frac{5}{-10} = -\frac{1}{2}$.

$$\therefore Z = \begin{bmatrix} 1 \\ -2 \\ -1/2 \end{bmatrix}$$

Now

$UX = Z$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{pmatrix} 1 \\ -2 \\ -1/2 \end{pmatrix}$$

using backward Substitution -

$x_1 + x_2 + x_3 = 1 \Rightarrow x_1 = 1 - \frac{1}{2} + \frac{1}{2} = 1$

$x_2 + 5x_3 = -2$ $\Rightarrow$ $x_2 = -2 - 5\cdot(-\frac{1}{2})$

$= -2 + \frac{5}{2}$

$x_3 = -1/2$

$= \frac{1}{2}$.

$$\therefore X = \begin{bmatrix} 1 \\ 1/2 \\ -1/2 \end{bmatrix}$$

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VII Triangularization method / LU decomposition:

Let A be the coefficient matrix of system ①

∴ In this method the coefficient matrix A of the system ① is decomposed or factorized into the product of a lower triangular matrix L and an upper triangular matrix U. (Jain / Iyengar / Jain)

we write the matrix A as

$$A = LU$$

where
$$L = \begin{bmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ l_{31} & l_{32} & l_{33} & \cdots & 0 \\ \cdots & \cdots & \cdots & & \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{bmatrix} \rightarrow \begin{bmatrix} l_{i1} & l_{i2} & \cdots & l_{in} \end{bmatrix} \begin{bmatrix} u_{1j} \\ u_{2j} \\ \vdots \\ u_{nj} \end{bmatrix}$$

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ \cdots & \cdots & \cdots & & \cdots \\ 0 & 0 & \cdots & \cdots & u_{nn} \end{bmatrix} \rightarrow$$

where $l_{ij} = 0 \quad j > i$
$u_{ij} = 0 \quad i > j$

using the matrix multiplication rule to multiply the matrices L and U and comparing the elements of the resulting matrix with those of A we obtain

$$l_{i1}u_{1j} + l_{i2}u_{2j} + l_{i3}u_{3j} + \cdots + l_{in}u_{nj} = a_{ij} \quad ② \qquad [j = 1(1)n.]$$

$$\text{where } l_{ij} = 0 \quad i < j$$
$$u_{ij} = 0 \quad i > j$$

The system of equation involves n^2+n unknowns. Thus, there are n parameters family of Soln. To produce a unique Soln it is convenient to choose either $u_{ii} = 1$ or $l_{ii} = 1$, $i = 1(1)n$.

when we chose

(i) $l_{ii} = 1$, the method is called Doolittle's method

(ii) $u_{ii} = 1$ " " " Crout's ".

when we take $u_{ii}=1$, $i=1(1)n$, the solution of the eqn ② may be written as

$$l_{ij} = a_{ij} - \sum_{k=1}^{j-1} l_{ik}u_{kj} \quad i \geq j$$

$$u_{ij} = a_{ij} - \sum_{k=1}^{i-1} \frac{l_{ik}u_{kj}}{l_{ii}} \quad i<j$$

$\left[l_{i1}u_{1j} + l_{i2}u_{2j} + \cdots + l_{ik}u_{kj} + \cdots + l_{in}u_{nj} = a_{ij} \right.$

$\Rightarrow l_{ik}u_{kj} = a_{ij} - \_\_\_$

$u_{ii} = 1.$

we note that the first column of the matrix L is iddentical with the first column of the matrix A.

That is $l_{i1} = a_{i1}$, $i = 1(1)n$.

$$u_{1j} = \frac{a_{1j}}{l_{11}}, \quad j = 2(1)n.$$

The first column of L and the first row of U have been determined. we can now proceed to determine the second column of L and the second row of U.

$$l_{i2} = a_{i2} - l_{i1}u_{12} \quad i = 2(1)n.$$

$$u_{2j} = \frac{(a_{2j} - l_{21}u_{1j})}{l_{22}}, \quad j = 3(1)n.$$

Next, we find the third column of L followed by the third row of U. Thus, for the relevant idices i and j, the elements are computed in the order

$$l_{i1}, u_{1j};\ l_{i2}, u_{2j},\ l_{i3}, u_{3j};\ \cdots\ l_{nn}.$$

Having determined the matrices L and U the system of equations (1) become

$$LUX = b.$$

$$\Rightarrow L(UX) = b \longrightarrow ③$$

we write (3) as the following two systems of eqn $UX = Z \rightarrow (4)$

$LZ = b \rightarrow (5)$

The unknown $z_1, z_2, \ldots z_n$ (4) are determined by forward substitution and the unknown $x_1, x_2 \ldots x_n$ in (5) are obtained by back substitution.

Alternatively, we find L^{-1} and U^{-1} to get

$$Z = L^{-1} b \quad \text{and} \quad X = U^{-1} Z$$

The inverse of A can also be determined from

$$A^{-1} = U^{-1} L^{-1}$$

<u>Note</u>: This method fails if any of the diagonal elements l_{ii} or u_{ii} is zero. The LU decomposition is guaranteed when the matrix A is positive definite. However, it is only a sufficient condition.