Date:- 23.09.20

Convergence Analysis of Iteration method

To discuss the convergence of Iterative method, and ~~to~~ study some theory about convergence, we first discuss some new concept about 'vector norms' and 'matrix Norms'.

I Vector Norm:

Def^n: The function $\|\cdot\|: \mathbb{R}^n \to \mathbb{R}$ is called 'Vector Norm' if for all $x, y \in \mathbb{R}^n$, $\forall \alpha \in \mathbb{R}$, the following properties hold:

(i) $\|x\| \geq 0$;

(ii) $\|x\| = 0$ iff $x = 0$.

(iii) $\|\alpha x\| = |\alpha| \, \|x\|$

(iv) $\|x + y\| \leq \|x\| + \|y\|$

The most commonly used norms are

(i) Absolute norm $\|x\| = \sum_{i=1}^{n} |x_i|$

(ii) Euclidian norm $\|x\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$

II Matrix Norm:

To measure the errors introduced during the Sol^n of a linear system, it will ~~[illegible]~~ be necessary to have a means of quantifying the "size" of a matrix. This is done using matrix norms.

Def^n: A matrix Norm is a function $\|\cdot\|: \mathbb{R}^{n \times n} \longrightarrow \mathbb{R}$ that, for all $A, B \in \mathbb{R}^{n \times n}$ and all $\alpha \in \mathbb{R}$, satisfies.

(i) $\|A\| \geq 0$

(ii) $\|A\| = 0$ iff $A = 0$

(iii) $\|\alpha A\| = |\alpha| \, \|A\|$

(iv) $\|A + B\| \leq \|A\| + \|B\|$

(v) $\|AB\| \leq \|A\| \, \|B\|$

Def^n: The "Spectral Radius" $\rho(A)$ of the matrix A is defined by

$$\rho(A) = \max_{\lambda \in \sigma(A)} |\lambda|$$

Where $\sigma(A)$ is the set of all spectrum or eigenvalue.

Most Important property is $\rho(A) \leq \|A\|$ for any natural norm.

VII Error Analysis of Iteration method

We know any problem of system of linear equation $Ax = b$, can be interpreted as root finding problem

$$Ax - b = 0.$$

and corresponding iteration formula is $x^{(k+1)} = Hx^{(k)} + c \longrightarrow (1)$

where H is called iteration matrix.

The analysis of the functional iteration scheme given by (1) boils down to four important Questions → (1) What conditions will guarantee a unique solution to the fixed point problem?

(2) under what conditions will the sequence generated by (1) converges to this unique fixed point?

(3) When the sequence generated by (1) converges, how quickly does it converges?

(4) What conditions must the matrix H and c satisfy in order for the fixed point problem to be consistent with the original rootfinding problem.

Theorem: Let A be an $n \times n$ matrix. Then the following statements are equivalent —

(i) $\rho(A) < 1$ ($\rho(A)$ is spectral radius of A)

(ii) $A^K \to 0$ as $K \to \infty$

(iii) $A^K x \to 0$ as $K \to \infty$ for any vector x.

$\Rightarrow$ (i) $\Leftrightarrow$ (ii)

Let $\rho(A) < 1$. ~~Since $\rho(A) \le \|A\|$~~

Now $\|A^K\| = \|A \cdot A \cdot \ldots A$ (K times)$\|$

$\le \|A\|\|A\|\|A\| \ldots \|A\|$ (K-times).

$= \|A\|^K$

$\therefore \|A^K\| \le \|A\|^K$.

$\therefore \lim\limits_{K \to \infty} A^K = 0$ iff $\|A\| < 1$

$\Rightarrow \lim\limits_{K \to \infty} A^K = 0$ iff $\rho(A) < 1$

(ii) $\Leftrightarrow$ (iii) [if $\lim A^K = 0$ then $\lim A^K x = 0$ $\forall x$, conversely it is true.

(iii) $\Leftrightarrow$ (i) [we see that $\lim A^K x = 0 \Leftrightarrow \lim\limits_{K \to 0} A^K = 0 \Leftrightarrow \rho(A) < 1$]

Th: The iteration scheme $x^{(K+1)} = Hx^{(K)} + c$ will converges for any choice of initial vector $x^{(0)}$ iff $\rho(H) < 1$

$\Rightarrow$ Let x^* denote the solution to the fixed point problem and define the iteration error vector $e^{(K)} = x^{(K)} - x^*$. Since x^* is fixed point so $x^* = Hx^* + c$.

$\therefore e^{(K+1)} = x^{(K+1)} - x^*$

$= \cancel{Hx^{(K)} + c - e^{(K)} - x^*}$

$= \cancel{Hx^{(K)} + c - e^{(K)} - Hx^* + c}$

$= Hx^{(K)} + c - Hx^* - c$

$= H[x^{(K)} - x^*] = He^{(K)}$

$\therefore$ we get $e^{(k+1)} = He^{(k)}$.

working backward through this eq^n we find.

$$e^{(k+1)} = He^{(k)}$$
$$= H(He^{(k-1)})$$
$$= H^2 e^{(k-1)}$$
$$= H^2(He^{(k-2)}) = H^3 e^{(k-2)}$$
$$= \dots$$
$$= H^{(k+1)} e^{(0)}$$

Ideally, $e^{(k+1)} \to 0$ as $k \to \infty$.

for any choice of initial vector $x^{(0)}$.

i.e, for any initial ~~vector~~ error vector $e^{(0)}$.

clearly $\lim_{k\to 0} e^{(k+1)} = 0 \Leftrightarrow \lim_{k\to 0} H^{(k+1)} e^{(0)} = 0$

$\Leftrightarrow e^{(0)} \lim_{k\to 0} H^{(k+1)} = 0$

$\Leftrightarrow \lim_{k\to 0} H^{(k+1)} = 0$

$\Leftrightarrow \rho(H) < 1$.

<u>Th</u>: The iteration scheme $x^{(k+1)} = Hx^{(k)} + c$ converges to a unique sol^n iff $\rho(H) \leq 1$, for any initial vector.

$\Rightarrow$ Let x be the unique sol^n

Then $x = Hx + c$.

$\Leftrightarrow x - Hx = c$

$\Leftrightarrow x(I - H) = c$

So x be the unique sol^n $\Leftrightarrow x(I-H) = c$

$\Leftrightarrow (I-H)$ is non singular

$\Leftrightarrow \rho(H) < 1$

Th: The iteration scheme $x^{(k+1)} = Hx^{(k)} + c$ converges fast to the exact root if $\rho(H)$ is very small.

$\Rightarrow$ For any initial approximation $x^{(0)} = 0$ (without loss of generality) we see that

$$\|e^{(k+1)}\| = \|He^{(k)}\|$$

$$\le \|H\| \, \|e^{k}\|$$

for any vector norm $\|\cdot\|$ and associated natural matrix norm.

$$x^{(1)} = Hx^{(0)} + c = c$$

$$x^{(2)} = Hx^{(1)} + c = \cancel{H(Hx^{(0)} + c) + c}$$

$$= \cancel{H^2 x^{(0)} + c(H+I)}$$

$$= Hc + c = (H+I)c$$

$$x^{(3)} = Hx^{(2)} + c = H(H+I)c + c$$

$$= (H^2 + H + I)c$$

$$\vdots$$

$$x^{(k)} = (H^{k-1} + H^{k-2} + \cdots + H + I)c.$$

Also $\|H\| < 1$

$$\|e^{(k)}\| \le \frac{\|H\|^k}{1 - \|H\|} \|x^{(1)} - x^{(0)}\|$$

$\therefore$ From this two inequality $\{x^{(k)}\}$ converges linearly with an asymptotic error constant that is less than or equal to $\|H\|$.
Also it can be shown that asymptotic error constant is equal to $\rho(H)$.
Thus the smaller the spectral radius of the iteration matrix, the faster the convergence of the corresponding iterative scheme.

(B)

[VI] The final preliminary issue to discuss is that of consistency. In order for the iteration defined by $x^{(k+1)} = Hx^{(k)} + c$ to be of any ~~partic~~ practical use, the solution of the fixed point problem $x^* = (I-H)^{-1}c$ must be identical to the soln of the original linear system $x^* = A^{-1}b$.

Hence, when constructing the fixed point problem from the linear system, we must be certain that H and c satisfy the relation

$$(I-H)^{-1}c = A^{-1}b.$$

Th: If A is a strictly diagonally dominant matrix, then the Jacobi iteration converges for any initial starting vector.

$\Rightarrow$ The Jacobi iteration scheme is given by

$$\rightarrow x^{(k+1)} = -D^{-1}(L+U)x^{(k)} + D^{-1}b$$

$$= -D^{-1}(A-D)x^{(k)} + D^{-1}b \quad [\because A = L+D+U,\ L+U = A-D]$$

$$= (-D^{-1}A + I)x^{(k)} + D^{-1}b.$$

$$= \underbrace{(I - D^{-1}A)}_{H} x^{(k)} + \underbrace{D^{-1}b}_{c}$$

The iteration scheme will be converges iff $\|(I - D^{-1}A)\| < 1 \rightarrow (1)$

if the $\|\cdot\|$ is a absolute row sum norm

i.e, $\|A\| = \frac{1}{|a_{ii}|}\sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|$

$\therefore$ So by (1) $\frac{1}{|a_{ii}|}\sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| < 1$

$\Leftrightarrow \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| < |a_{ii}| \Leftrightarrow A$ is strictly diagonally dominant