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Interpolation and Approximation.

In this chapter, we consider the problem of approximating a given function by a class of simpler functions, mainly polynomial.

There are two main uses of interpolation or interpolating polynomials.

The first is → reconstructing the function f(x) when it is not given explicitly and only the values of f(x) and/or its certain order derivatives at a set of points, called nodes, tabular points or arguments are known.

The second is ⟶ to replace the function f(x) by an interpolating polynomial P(x) so that many common operations such as determination of roots, differentiation and integration etc. which are intended for the function f(x) may be performed using P(x).

In approximation, we measure the derivation of the given function f(x) from the approximating polynomial P(x) for all values of x over a given interval [a, b].

we first discuss the methods to construct the interpolating polynomials P(x) to a given function f(x).

Defn: A polynomial $p(x)$ is called an interpolating polynomial if the values of $p(x)$ coincide with those of $f(x)$ at one or more tabular points.

That is, the mathematical problem ~~started~~ actually gives ries to two different areas of study : interpolation and approximation.

There are many different types of interpolation depending upon the class of functions from $p(x)$ is selected.

The most common forms of interpolation are

(1) polynomial interpolation.
(2) piecewise polynomial (spline) interpolation
(3) rational interpolation.
(4) trigonometric interpolation.
(5) exponential interpolation.

The focus of this chapter will be on polynomial interpolation.

There are ~~three~~ major reasons for focusing on interpolation by polynomial and piecewise polynomials.

(1) The polynomial $p(x)$ can be evaluated very efficiently using the synthetic division algorithm.

(2) Derivatives and integration of polynomials are easy to compute and are still polynomials

(3) polynomial satisfy uniform approximation property.

Theorem: Let f be continuous on the closed interval $[a, b]$. Given any $\varepsilon > 0$, $\exists$ a polynomial P such that

$$\|f - P\|_\infty \equiv \max_{x \in [a,b]} |f(x) - p(x)| < \varepsilon$$

(Weierstrass Approximation Theorem)

This interpolating polynomial is disguise in none other than the Taylor polynomial.

Taylor Series

If the polynomial $p(x)$ is written as the Taylor's expansion, for the function $f(x)$ about a point $x_0 \in [a, b]$, in the form.

$$p(x) \equiv f(x_0) + (x - x_0) f'(x_0) + \frac{1}{2!}(x - x_0)^2 f''(x_0) + \cdots + \frac{1}{n!}(x - x_0)^n f^{(n)}(x_0) \longrightarrow (1)$$

then, $p(x)$ ~~...~~ may be regarded as an interpolating polynomial of degree n, Satisfying the conditions $p^{(k)}(x_0) = f^{(k)}(x_0)$, $k = 0, 1, \ldots n$.

The term $R_n = \frac{1}{(n+1)!}(x - x_0)^{n+1} f^{(n+1)}(\xi)$, $x_0 < \xi < x$

which has been neglected in (1), is called the remainder or the truncation error.

The number of terms to be included in (1) is called the remainder or the truncation error.

If this error is $\varepsilon > 0$ and the series is truncated at the term $f^{(n)}(x_0)$, then

$$\frac{1}{(n+1)!} |x - x_0|^{n+1} |f^{(n+1)}(x)| \leq \varepsilon$$

or $$\frac{1}{(n+1)!} |x - x_0|^{n+1} M_{n+1} \leq \varepsilon$$

where $M_{n+1} = \max_{a \leq x \leq b} |f^{(n+1)}(x)|$

Assume that the value of M_{n+1} or its estimate is available.

For a given ε and x (2) will determine n, and if n and x are prescribed, it will determine ε. When both n and ε are given, it will give an upper bound on $(x-x_0)$, that is, it will given an interval about x_0 in which this Taylor's polynomial approximates $f(x)$ to the prescribed accuracy.

<u>Ex!</u> obtain the Taylor series approximation about $x=1$, upto second degree terms for the function $f(x)=\frac{1}{1+x^2}$. Find a bound on the error if this approximation is to be used in (10) $[1, 1.4]$

We have $f(x)=\frac{1}{1+x^2}$, $f'(x)=-\frac{2x}{(1+x^2)^2}$

$$f''(x)=-\frac{2(1-3x^2)}{(1+x^2)^3}, \quad f'''(x)=-\frac{24x(x^2-1)}{(1+x^2)^4}$$

and $f(1)=\frac{1}{2}$, $f'(1)=-\frac{1}{2}$, $f''(1)=\frac{1}{2}$.

The Taylor series approximation is given by

$$f(x)=f(1)+(x-1)f'(1)+\frac{1}{2}(x-1)^2 f''(1)$$

$$=\frac{1}{2}-\frac{1}{2}(x-1)+\frac{1}{4}(x-1)^2$$

The error bound is given by

$$|R_2|\le \frac{(x-1)^3}{3!}M_3 \quad \text{where } M_3=\max_{1\le x\le 1.4}|f'''(x)|$$

We obtain $M_3=\frac{24(1.4)(0.96)}{16}=2.016$

$\therefore$ Therefore $|R_2|\le \frac{(x-1)^3}{6}(2.016)=0.336(x-1)^3$

Maximum absolute error occurs at $x=1.4$ is 0.0215