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Interpolation and Approximation

From previous leacture we get,

$p(x)$ is a interpolating polynomial and

$p(x) = a_0 + a_1 x + \dots + a_n x^n$, where $a_0, a_1, \dots a_n$ are obtain from the system of eqn

$$a_0 + a_1 x_0 + a_2 x_0^2 + \dots + a_n x_0^n = f(x_0)$$
$$a_0 + a_1 x_1 + a_2 x_1^2 + \dots + a_n x_1^n = f(x_1)$$
$$a_0 + a_1 x_n + a_2 x_n^2 + \dots + a_n x_n^n = f(x_n)$$

Now we show that the polynomial $p(x)$ obtained above system is unique.

Proof: Let $f(x)$ be a function which is continuous on $[a, b]$ and f is known for the poins $a \le x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n \le b$ and $p(x), p^*(x)$ are two interpolating polynomial of $f(x)$. which also satisfies

$$\left.\begin{array}{l} p(x_i) = f(x_i) \\ p^*(x_i) = f(x_i) \end{array}\right\} i = 0, 1, \dots n.$$

Consider the polynomial

$$g(x) = p(x) - p^*(x).$$

Since $p(x)$ and $p^*(x)$ are both polynomials of degree $\le n$, $g(x)$ is also a polynomial of degree $\le n$ satisfying the conditions

$$g(x_i) = p(x_i) - p^*(x_i) = 0, \quad i = 0, 1, 2 \dots n.$$

Therefore, $g(x)$ is a polynomial of degree $\le n$ which has $n+1$ distinct root $x_0, x_1, x_2, \dots, x_n$. This implies that $g(x) \equiv 0$ because a polynomial $g(x)$ of degree n has exactly n roots, real or complex.

$\therefore \; p^*(x) = p(x)$.

VII Thus, the interpolating polynomial obtained in two different ways may be different in form but are identical otherwise.

Depending on its form, the polynomial is called either the Lagrange interpolating polynomial or the Newton divided difference interpolating polynomial.

We discuss now interpolations of various degrees:

VIII Linear Interpolation: we assume that we are given an $[a,b]$ a function f which is continuous on $[a,b]$. Further we assume that we have $(n+1)$ distincts $a \le x_0 < x_1 < x_2 < \cdots < x_n \le b$ of $[a,b]$ and that the values of a function $f(x)$ are known at these points. We seek to find the polynomial $P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

For linear interpolation, $n=1$, i.e., we want to determine a polynomial

$$P(x) = a_1 x + a_0 \longrightarrow (1)$$

where a_0 and a_1 are arbitrary constants which satisfies the interpolating conditions $f(x_0) = P(x_0)$ and $f(x_1) = P(x_1)$.

We have
$$\left.\begin{aligned} f(x_0) &= P(x_0) = a_1 x_0 + a_0 \\ f(x_1) &= P(x_1) = a_1 x_1 + a_0 \end{aligned}\right\} \longrightarrow (2)$$

Eliminating a_0 and a_1 from (1) and (2), we obtain the required linear interpolating polynomial as

$$\begin{vmatrix} P(x) & x & 1 \\ f(x_0) & x_0 & 1 \\ f(x_1) & x_1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow P(x)\begin{vmatrix} x_0 & 1 \\ x_1 & 1 \end{vmatrix} - f(x_0)\begin{vmatrix} x & 1 \\ x_1 & 1 \end{vmatrix} + f(x_1)\begin{vmatrix} x & 1 \\ x_0 & 1 \end{vmatrix} = 0$$

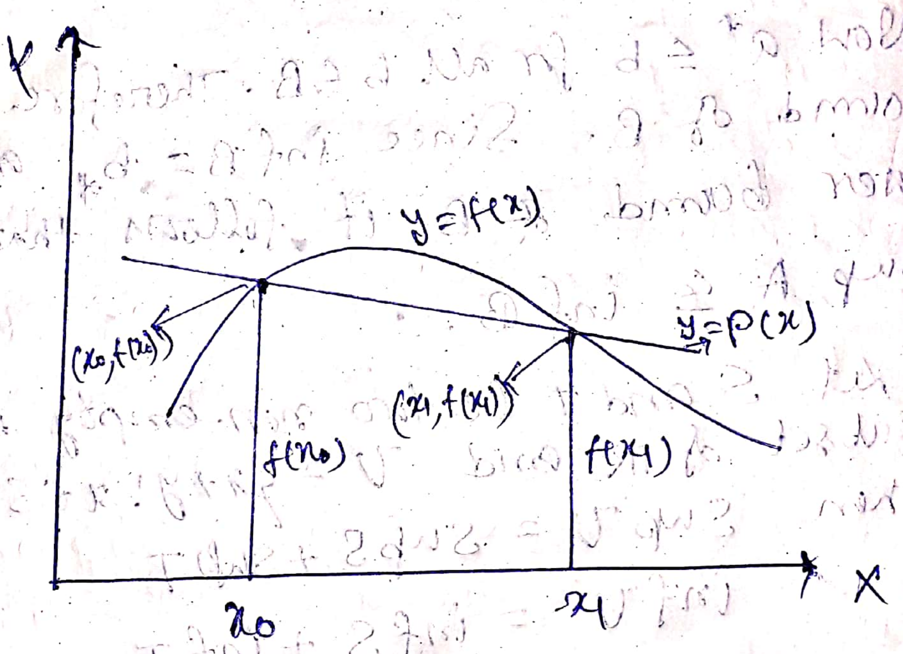
$$\Rightarrow P(x)(x_0 - x_1) - f(x_0)(x - x_1) + f(x_1)(x - x_0) = 0$$

$$\Rightarrow P(x)(x_0 - x_1) = (x - x_1) f(x_0) - f(x_1)(x - x_0)$$

$$\Rightarrow P(x) = \frac{(x - x_1)}{x_0 - x_1} f(x_0) - \frac{(x - x_0)}{x_0 - x_1} f(x_1)$$

$$= \frac{(x - x_1)}{(x_0 - x_1)} f(x_0) + \frac{(x - x_0)}{(x_1 - x_0)} f(x_1).$$

Graphically



Linear interpolation