Date- 08.10.20

Properties of the Suprimum and infimum

(1) Let A, B be bounded subset of $\mathbb{R}$ such that $x \in A, y \in B \Rightarrow x \leq y$; then $\sup A \leq \inf B$.

$\Rightarrow$ Since A, B are non-empty bounded subsets of $\mathbb{R}$, $\sup A$ and $\inf B$ exist. Let $\sup A = a^*$
$\inf B = b_*$.

Let $b \in B$. Then $x \in A \Rightarrow a \leq b$, This shows that b is an upper bound of A. Since $\sup A = a^*$ and b is an upper bound of A it follows that $a^* \leq b$.

Now $a^* \leq b$ for all $b \in B$. Therefore a^* is a lower bound of B. Since $\inf B = b_*$ and a^* is a lower bound of B it follows that $a^* \leq b_*$ i.e.

$\sup A \leq \inf B$.

(2) Let S and T be two non-empty bounded subset of $\mathbb{R}$ and $U = \{x+y : x \in S, y \in S\}$
Then $\sup U = \sup S + \sup T$
$\inf U = \inf S + \inf T$.

$\Rightarrow$ Let $s^* = \sup S$,
$t^* = \sup T$.
Now we show that $s^* + t^*$ is upper bound of U

Let $s + t \in U \because s \in S, t \in T$.
Since $s^* = \sup S$ and $t^* = \sup T$ so
$s \leq s^*, t \leq t^* \Rightarrow s + t \leq s^* + t^*$
Which is true for any $s \in S, t \in T$.
So $\forall s + t \in U$, $s + t \leq s^* + t^*$

$\therefore$ $s^* + t^*$ is upper bound of U

Let u be any upper bound of U.

Then $s+t \le u$ $(\forall s \in S, t \in T)$ $\forall s+t \in U$.

$\Rightarrow t \le u - s$.

$\Rightarrow t \le u-s$ $\forall t \in T$.

$\Rightarrow$ $u-s$ is upper bound of T.

Since t^* ~~is~~ is $\sup T$ so.

$t^* \le u-s$ $\forall s \in S$. ($\because t^*$ is least upperbound)

$\therefore$ ~~Similar $s^* \le u-t$ $\forall t \in T$~~

$\Rightarrow$

~~Again $u-s$ is upper bound of~~

Again $s \le u - t^*$ $\forall s \in S$.

$\Rightarrow$ $u - t^*$ is upper bound of S

$\Rightarrow$ $s^* \le u - t^*$ $\because$ [$s^* = \sup S$, so s^* is least upper bound].

$\Rightarrow$ $s^* + t^* \le u$.

$\therefore$ $s^* + t^*$ is least upper bound.

$\therefore$ $s^* + t^* = \sup U$

$\therefore$ $\sup U = \sup S + \sup T$.

Second proof is similar as above