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Newton's Divided Difference interpolation

For linear interpolation, let f is a continuous function define in $[a, b]$ and f is known for x_0 & x_1 and we want to find a poly^n $P(x) = a_0 + a_1 x$. where a_0 and a_1 are arbitrary constant, which satisfies the interpolation condition $f(x_0) = P(x_0)$

$$f(x_1) = P(x_1)$$

Then

$$a_1 x_0 + a_0 = f(x_0)$$

$$a_1 x_1 + a_0 = f(x_1)$$

Eliminating a_0 and a_1, we obtain the required linear interpolating pol^n -

$$\begin{vmatrix} P(x) & x & 1 \\ f(x_0) & x_0 & 1 \\ f(x_1) & x_1 & 1 \end{vmatrix} = 0$$

we now, expand the determinant in terms of the first row and get

$$P(x) = f(x_0) + (x - x_0) \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$= f(x_0) + (x - x_0) f[x_0, x_1]$$

where $\frac{f(x_1) - f(x_0)}{x_1 - x_0} = f[x_0, x_1]$

The ratio $f[x_0, x_1]$ is called the first divided difference of $f(x)$ relative to x_0 and x_1.
We may write as

$$\frac{p(x) - f(x_0)}{x - x_0} = f[x_0, x_1] \longrightarrow ②$$

eqn ① and ② are called the linear Newton interpolating polynomial with divided differences.

☒ Now we generalize, the Newton divided difference
we know $f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$= \frac{1}{(x_2 - x_0)}\left[\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}\right]$$

$$= \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)} - \frac{f(x_1)}{(x_2 - x_0)}\left[\frac{1}{x_2 - x_1} + \frac{1}{x_1 - x_0}\right] + \frac{f(x_0)}{(x_2 - x_0)(x_1 - x_0)}.$$

$$= \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)} - \frac{f(x_1)\,(x_1 - x_0 + x_2 - x_1)}{(x_2 - x_0)(x_2 - x_1)(x_1 - x_0)} + \frac{f(x_0)}{(x_2 - x_0)(x_1 - x_0)}$$

$$= \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)}.$$

$$\therefore f[x_0, x_1, \ldots, x_{k-1}, x_k] = \frac{f[x_1, x_2, \ldots, x_k] - f[x_0, x_1, \ldots, x_{k-1}]}{x_k - x_0}$$

$$k = 3, 4, \ldots n.$$

In terms of functions values, the n-th divided difference can be written as

$$f[x_0, x_1, x_2, \ldots, x_n] = \sum_{i=0}^{n} \frac{f(x_i)}{\prod_{\substack{j=0 \\ i \neq j}} (x_i - x_j)}$$

using this divided difference concept we can calculate a interpolating polynomial.

Clearly $f[x_0, x_1] = f[x_1, x_0]$.

$f[x_0, x_1, x_2] = f[x_2, x_1, x_0]$.

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$f[x_0, x_1, \ldots x_n] = f[x_n, x_{n-1}, \ldots x_0]$.

Now $P_n(x)$ be a interpolating poln of $f(x)$ on $[a, b]$ and $f(x)$ is known for $n+1$ argument $a \leq x_0 \leq x_1 \leq x_2 \leq \ldots \leq x_n \leq b$.

and let $P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \ldots + (x-x_0)(x-x_1)\ldots(x-x_{n-1})a_n$.

Substituting successively $x = x_0, x = x_1, \ldots, x = x_n$ we obtain

$P_n(x_0) = a_0 = f(x_0)$.

$P_n(x_1) = a_0 + a_1(x_1 - x_0) = f(x_1)$.

$\Rightarrow a_1(x_1 - x_0) = f(x_1) - f(x_0)$

$\Rightarrow a_1 = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0} = f[x_1, x_0]$

$P_n(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$.

$\Rightarrow f(x_2) = f(x_0) + f[x_1, x_0](x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$

$\Rightarrow a_2(x_2 - x_0)(x_2 - x_1)$

$= f(x_2) - f(x_0) - f[x_1, x_0](x_2 - x_0)$

$$\Rightarrow a_2 = \frac{f(x_2) - f(x_0)}{(x_2 - x_0)(x_2 - x_1)} + \frac{f[x_1, x_0]\cancel{(x_2 - x_0)}}{\cancel{(x_2 - x_0)}(x_2 - x_1)}$$

$$= \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)} + \frac{f(x_0)}{(x_0 - x_2)(x_2 - x_1)} + \frac{f(x_1) - f(x_0)}{(x_1 - x_0)(x_2 - x_1)}$$

$$= \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)} + \frac{f(x_0)}{(x_2 - x_1)}\left[\frac{1}{x_0 - x_2} + \frac{1}{x_1 - x_0}\right] + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)}$$

$$= \frac{f(x_0)}{(x_0 - x_2)(x_0 - x_1)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)} = f[x_0, x_1, x_2]$$

Using induction, we can prove that

$$a_n = f[x_0, x_1, \ldots x_n].$$

The divided difference interpolating polynomial becomes

$$P_n(x) = f[x_0] + (x-x_0) f[x_0,x_1] + (x-x_0)(x-x_1) f[x_0,x_1,x_2] + \cdots + (x-x_0)(x-x_1)\cdots(x-x_{n-1}) f[x_0,x_1,\ldots x_n].$$

Note: Since the interpolating polynomial is unique, Lagrange and divided difference polynomials are two different forms of the same polynomial.