CC-03

Unit-1

Countable sets

Defn (a) A set S is said to be countably infinite / (denumerable) if there exists a bijection of $\mathbb{N}$ onto S.

(b) A set S is said to be countable iff it is either finite or countably infinite.

(c) A set S is said to be uncountable if it is not countable $\Rightarrow$ there is no bijection from $\mathbb{N}$ to S.

Note: (i) A set S is countably infinite iff $\exists\ f: \mathbb{N} \to S$ which is bijective.

$\Rightarrow$ A set S is countably infinite iff $\exists\ g: S \to \mathbb{N}$ which is bijective.

(ii) Let S_1 be a countably infinite set. Then S is countable infinite iff $\exists\ f: S \to S_1$ which is bijective.

(iii) If A is countable iff $A \sim B \subseteq \mathbb{N}$

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Examples:

(i) Set of natural number $\mathbb{N}$ is countably infinite set.

(2) The set $E = \{2n : n \in \mathbb{N}\}$ of even numbers is denumerable.

Proof: Let us define $f : \mathbb{N} \to E$ by $f(n) = 2n \;\forall n \in \mathbb{N}$

Clearly $\forall n_1, n_2 \in \mathbb{N}$, $f(n_1) = f(n_2)$

$$\Rightarrow 2n_1 = 2n_2$$

$$\Rightarrow n_1 = n_2$$

$\therefore$ f is one-one.

by the defination $\forall\, 2n \in E \;\exists\; n \in \mathbb{N}$ s.t $f(n) = 2n$. So f is onto.

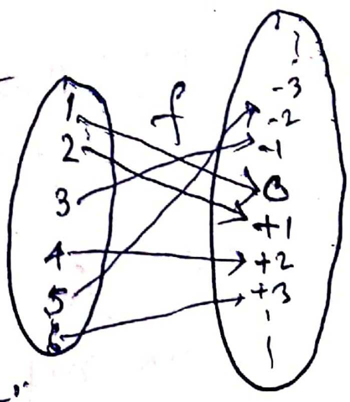
So $\exists$ bijection $f : \mathbb{N} \to E$. Hence E is countable infinite or denumerable.

Similarly, the set $O = \{2n-1 : n \in \mathbb{N}\}$ of odd natural number is denumarable.

(3) The set $\mathbb{Z}$ of all integers is denumarable.

Proof: Let us define $f : \mathbb{N} \to \mathbb{Z}$ by

$$f(n) = \begin{cases} 0 & \text{if } n = 1 \\ n & \text{if } n \text{ is even} \\ -n & \text{if } n \text{ is odd.} \end{cases}$$



Then clearly f is bijective.

$\therefore \mathbb{Z}$ is denumarable.

Th: The union of two disjoint countable set is countable.

proof: Let A and B are two countable set

$\Rightarrow$ A and B are finite or countable infinite.

case-i: If A and B are finite then $A \cup B$ is finite.

case-ii: If A and B are countable infinite then

$$A = \{a_1, a_2, \ldots\} \quad B = \{b_1, b_2 \ldots\}$$

$$A \cup B = \{a_1, a_2, \ldots, b_1, b_2, \ldots\}$$ which is also countable infinite.

(1) Th: Suppose that S and T are set and that $T \subseteq S$

(a) if S is a countable set, then T is countable.

(b) if T is an uncountable set, then S is an uncountable set.

proof: (a) Since S is countable infinite set so $\exists$ a bijective mapping $f: \mathbb{N} \to S$

Since $T \subseteq S$ so $f: \mathbb{N} \to T$ is also bijective.

$\therefore$ T is countable infinite.

Again if S is finite then T is also finite.

Therefor T is countable.

(b) Let S is countable then $T \subseteq S \Rightarrow$ T is countable which is contradiction. So S is uncountable.

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(2) **Th:** The following are equivalent

1. A is countable infinite.
2. $\exists$ a subset B of $\mathbb{N}$ and a mapping $f: B \to A$ which is onto.
3. $\exists$ a subset C of $\mathbb{N}$ and a mapping $g: A \to C$ which is one-one.

proof

Clearly $\boxed{1 \Rightarrow 2}$ (you can take $B = \mathbb{N}$ and $C = \mathbb{N}$).
$\boxed{1 \Rightarrow 3}$

Now we show $\boxed{3 \Rightarrow 1}$

Let $\exists$ a subset C of $\mathbb{N}$ and a mapping $g: A \to C$ which is one-one.

Then $g(A) \subseteq C$.

and $g: A \to g(A)$ is onto.

Therefore $g: A \to g(A) \subseteq C \subseteq \mathbb{N}$ is one-one and onto.

$\therefore$ A is countable infinite.

$\therefore \boxed{3 \Rightarrow 1}$

③ **Th:** The set $\mathbb{N}\times\mathbb{N}$ is denumerable.

Proof: Let us define a mapping

$$f: \mathbb{N}\times\mathbb{N} \to \mathbb{N} \text{ by } f(n,m) = 2^n \cdot 3^m$$

then we can show that f is bijective.

Therefore $\mathbb{N}\times\mathbb{N}$ is denumerable. (Home work)

④ **Th:** The set $\mathbb{Q}^+$ of positive rational number is denumerable.

Proof: Let us define $f: \mathbb{N}\times\mathbb{N} \to \mathbb{Q}^+$ by

$$f(m,n) = \frac{m}{n}.$$

then f is bijective (Home work).

Similarly $\mathbb{Q}^-$ of ~~positive~~ negative rational number is denumerable.

⑤ **Th:** The set $\mathbb{Q}$ of all rational number is denumerable.

Proof: clearly $\mathbb{Q} = \mathbb{Q}^+ \cup \{0\} \cup \mathbb{Q}^-$

Since $\mathbb{Q}^+ \cup \{0\}$ is union of two denumerable set so $\mathbb{Q}^+ \cup \{0\} = A$ is denumerable.

Again $\mathbb{Q} = A \cup \mathbb{Q}^-$ is also denumerable.

Hence $\mathbb{Q}$ is denumerable.

⑥ **Th:** If A_m is a countable set for each $m \in \mathbb{N}$, then the union $A = \bigcup_{m=1}^{\infty} A_m$ is countable.

Proof: For each $m \in \mathbb{N}$, let $\varphi_m : \mathbb{N} \to A_m$ be a bijection [$\because \forall m$, A_m are countable].

Let $\psi: \mathbb{N}\times\mathbb{N} \to A$ define by $\psi(m,n) = \varphi_m(n)$.

We only show that ψ is onto (Th: 2)

Let $a \in A$, $\exists\, m \in \mathbb{N}$ s.t $a \in A_m$, since φ_m is bijective so $\exists\, n \in \mathbb{N}$ s.t $\varphi_m(n) = a \Rightarrow a = \psi(m,n)$

$\Rightarrow \psi$ is onto.

Then by theorem ② A is countable

(7) Th: $(0,1)$ is uncountable set. (H T).

(2)

(8) Th: $\mathbb{R}$ is uncountable set (H.T).

Proof: Since $(0,1) \sim \mathbb{R}$ and $(0,1)$ is uncountable so $\mathbb{R}$ is uncountable.

Note:

A

Countable set

(i) $A \sim \mathbb{N}$.

(ii) A is finite.

(iii) $|A| = |\mathbb{N}| = \aleph_0$ (aleph not).

Ex: $\boxed{\mathbb{N}, \mathbb{Q}, \mathbb{Z}, \mathbb{N}\times\mathbb{N}, \mathbb{N}\times\mathbb{N}\times\cdots\times\mathbb{N}, E, \phi}$

Uncountable.

(i) $A \sim \mathbb{R}$

(ii) $|A| = |\mathbb{R}| = c$

(~~continuum~~

Ex: $\boxed{\mathbb{R}, (0,1), [0,1], (a,b), [a,b], \mathbb{R}^n, \mathbb{Q}^c, \mathbb{C}, P(\mathbb{N})}$

$\boxed{\text{Results}}$

(i) Uncountable − countable = uncountable.

Proof: Let A and B are two set if A is uncountable and B is countable $B \subseteq A$.

Then $A = B \cup B^c$ where $B^c = A \setminus B$.

~~...~~

We show that B^c is uncountable.

if B^c is ~~not~~ countable then $A = B \cup B^c$ become countable which is contradiction. $\because$ ~~...~~t.

$\therefore B^c = A \setminus B$ is uncountable.

(2) Uncountable − uncountable = may be countable or countable.

Proof: (i) $\mathbb{R}$ is uncountable; Then $\mathbb{R} - \mathbb{R} = \phi$ (countable)

(ii) $\mathbb{R} - \mathbb{Q}^c = \mathbb{Q}$ (which is countable).

onto function! Let $A \neq \phi$, $B \neq \phi$ and $f: A \to B$ ... function called onto function $\forall y \in B$, $\exists x \in A$ s.t $f(x) = y$ i.e, $f(A) = B$.

Note: (i) if $f: A \to B$ is a onto function ~~then~~ $\Rightarrow |A| \geq |B|$.

(ii) if $|A| < |B| \Rightarrow \nexists$ any onto function $f: A \to B$.

(iii) if $|A| \geq |B| \Rightarrow \exists f: A \to B$ which is onto.

(iv) Number of onto function from A to B

$$= \sum_{r=0}^{m-r} {}^{m}C_r (m-r)^n (-1)^r$$

Q: if $|A| = \{1, 2, 3 \cdots m\}$.

$|B| = \{a, b\}$.

Then number of onto function ??

Th prob: Show that $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$

proof: Let $f: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ define by $f(m,n) = 2^m 3^n$.

Then f is one-one and onto.

Since f is one-one so $|\mathbb{N} \times \mathbb{N}| \leq |\mathbb{N}| \to$ (i)

Again $f^{-1}: \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ is also bijection so f^{-1} is one-one.

$$|\mathbb{N}| \leq |\mathbb{N} \times \mathbb{N}| \to \text{(ii)}$$

$\therefore \quad |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$

Note: (1) $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}|$. (in similar way).

(2) if A and B are two nonempty set and $\exists$ a one-one, onto mapping $f: A \to B$ then $|A| = |B|$. (H T).

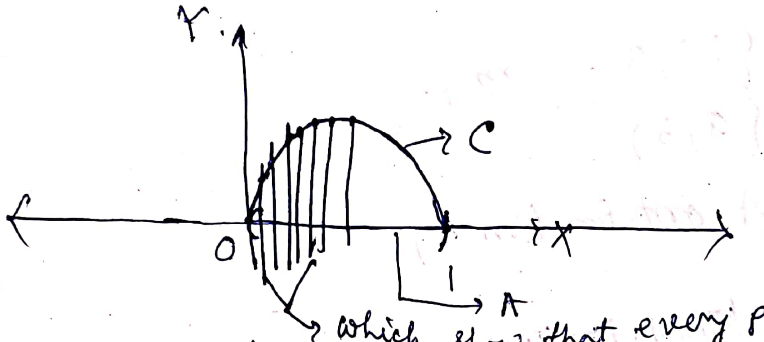
Def^n: Two set A an[d] ... similar iff $\exists$ a bijective mapping $f: A \to B$, which is denoted by $\mathbb{N} \sim \mathbb{N}\times\mathbb{N}\times\mathbb{N}$.

Ex: $(0,1) \sim \mathbb{R}$.

Proof: Let $A = (0,1)$. $B = \mathbb{R}$.

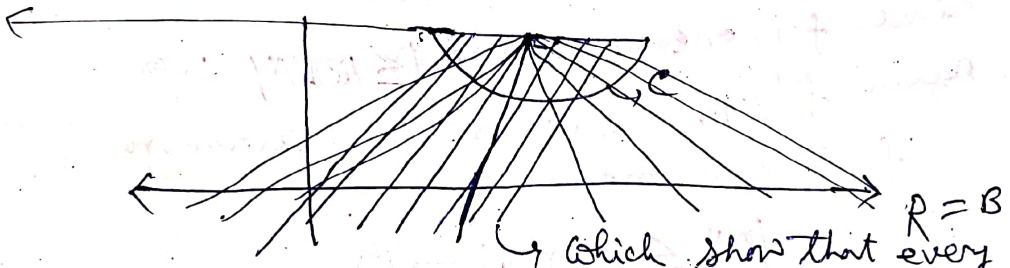
Let C = semi circle.

We shall show that $A \sim C$, $C \sim B$. then we can say that $A \sim B$.



which show that every point of C related with A.

$\therefore A \sim C$



which show that every point of C related with B.

$\therefore A \sim C$, $C \sim B$ then $A \sim B$.

$\Rightarrow (0,1) \sim \mathbb{R}$.

Ex: $\overset{A}{(0,1)} \sim (0,1)\times(0,1) \to B =$

Proof: Let $f: A \to B$ define by where $A = (0,1)$, $B = (0,1)\times(0,1)$.

$f(x) = (x, 1/a)$., $(a \neq 0)$

For one-one: Let $x_1, x_2 \in A$, then $f(x_1) = f(x_2)$

$\Rightarrow (x_1, 1/a) = (x_2, 1/a)$.

$\Rightarrow x_1 = x_2$

$\Rightarrow f$ is one-one.

$|A| \leq |B|$. $\Rightarrow |(0,1)| \leq |(0,1) \times (0,1)| \rightarrow ①$

Let us define $g : B \rightarrow A$. for defining

let $x = 0.x_1 x_2 \ldots x_m \ldots \in (0,1)$

$y = 0.y_1 y_2 \ldots y_n \ldots \in (0,1)$

where $x_i, y_i = \{0,1,2 \ldots 9\}$ $\forall i$

$z = 0.x_1 y_1 x_2 y_2 \ldots \in (0,1)$.

Now $g(x,y) = z$ $\forall (x,y) \in (0,1) \times (0,1) = B$.

<u>For one-one</u>:

$x = 0.x_1 x_2 \ldots x_m \ldots$

$y = 0.y_1 y_2 \ldots y_n \ldots$

$x' = 0.x_2 x_1 \ldots x_m \ldots$

$y' = 0.y_2 y_1 \ldots$

Then $(x,y) \neq (x',y')$.

$g(x,y) = 0.x_1 y_1 x_2 y_2 \ldots$

$g(x'y') = 0.x_2 y_2 x_1 x_1 \ldots$

clearly $g(x,y) \neq g(x',y')$

$\therefore g$ is one-one.

$\therefore |B| \leq |A|$.

$\Rightarrow |(0,1) \times (0,1)| \leq |(0,1)|. \rightarrow ②$

$\therefore$ From ① and ②

$|(0,1)| = |(0,1) \times (0,1)|$.

<u>Note</u>: Similarly $|(0,1)| = |(0,1) \times (0,1) \times \ldots \times (0,1)|$

$\therefore (0,1) \sim (0,1) \times (0,1] \times \ldots \times (0,1)$

(D) Again $(0,1) \sim R$ and $(0,1) \sim (0,1) \times (0,1)$

$\Rightarrow \mathbb{R} \sim \mathbb{R} \times R$

$\Rightarrow R \sim \mathbb{R}^2$

Note: (i) $R \sim \mathbb{R}^n \sim \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ (n times)

$\Rightarrow R \sim \mathbb{R}^n$.

(ii) $[0,1] = (0,1) \cup \{1\}$.

Therefor: $(0,1) \sim (0,1) \times (0,1)$

$\Rightarrow (0,1) \cup \{1\} \sim \left((0,1) \cup \{1\}\right) \times \left((0,1) \cup \{1\}\right)$

$\Rightarrow [0,1] \sim [0,1] \times [0,1]$.

Similarly $[0,1] \sim [0,1] \times [0,1] \times \cdots \times [0,1]$.

$\therefore [0,1] \sim [0,1] \times [0,1] \times \cdots \times [0,1]$.

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(3) Set of all disjoint open intervals is countable.
(H.T).

(4) set of all disjoint close intervals is alway uncountable.

(5) $\underbrace{\aleph_0 + \aleph_0 + \aleph_0 + \cdots + \aleph_0}_{n \text{ times}} = \aleph_0$

(6) $n\aleph_0 = \aleph_0$

(7) $c + c + \cdots + \cdots c = c$

(8) $2^{\aleph_0} = c, \quad \aleph_0^{\aleph_0} = c$

(9) $2^c = \aleph_0^c = c^c$

(10) $c < 2^c$

(11) $\aleph_0 < c$

(12) $c\aleph_0 = c$

Problem (V.V.I)

(countablity and uncountablity)

① Let A and B be two non empty set then if $f: A \to B$, verify the following statement

$\longrightarrow$

ⓐ A countable $\Rightarrow$ $f(A)$ countable.

ⓑ $f(A)$ countable $\Rightarrow$ A is countable.

ⓒ $f(A)$ uncountable $\Rightarrow$ A is uncountable

proof: ⓐ Since A is countable so $\exists$ $g: \mathbb{N} \to A$ which is bijection: Also $f: A \to B$. then $f: A \to f(A)$ is onto. $\therefore$ $g \circ f: \mathbb{N} \to f(A)$ be a onto mapping Therefore. $\exists$ a onto. $g \circ f: \mathbb{N} \to f(A)$ and $\mathbb{N}$ is countable so $f(A)$ is countable.

$\therefore$ $\boxed{\text{(a) is true.}}$

ⓑ Let $A = \mathbb{R}$ and $B = \{c\}$. we define a mapping $f: A \to B$ by $f(x) = c \ \forall x \in A$. i.e, f is constant mapping.

Then clearly B is countable i.e $f(A)$ is countable but $A = \mathbb{R}$ is uncountable. So $\boxed{\text{(b) is not true}}$

ⓒ Since $f(A) \subseteq B$ and $f(A)$ is uncountable so B must be uncountable. Then A is must be uncountable. if not i.e, A is countable then by A, $f(A)$ is countable but $f(A)$ is uncountable. so

$\therefore$ A is always uncountable.

$\therefore$ $\boxed{\text{(c) is true}}$

Q) Which of the following are true (varify).

(a) Every subset of $\mathbb{R}$ containing a non-empty open interval is uncountable.

(b) Every subset of $\mathbb{R}$ containing $\mathbb{Q}^c$ is similar to $\mathbb{R}$.

(c) $\forall n \in \mathbb{N}$, $\mathbb{R}^n$ similar to $\mathbb{R}$.

(d) $\forall n \in \mathbb{N}$, $(\mathbb{Q}^c)^n$ similar to $\mathbb{R}$.

Proof: (d) $(\mathbb{Q}^c)^n = \mathbb{Q}^c \times \mathbb{Q}^c \times \cdots \times \mathbb{Q}^c \sim \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$

$\Rightarrow (\mathbb{Q}^c)^n \sim \mathbb{R}^n \sim \mathbb{R}$.

(d) true.

Ans: (a, b, c, d) true.

(3)

$A = \{ f : [0,1] \to \mathbb{R} \} : f$ is a function, then

test the following statement.

(a) A is uncountable.

(b) A is similar to $\mathbb{R}$.

(c) A is similar to $P(\mathbb{R})$.

(d) A is similar to $P(\mathbb{N})$.

(e) cardinality of A is c^c

Proof: We know that if $|A| = m$, $|B| = n$. then number of mapping $f: A \to B$ is n^m.

Therefor. from our problem

$|[0,1]| = c$ $[\because [0,1] \sim \mathbb{R}]$.

$|\mathbb{R}| = c$.

$\therefore$ Number of mapping from $[0,1]$ to $\mathbb{R}$ are c^c

$\therefore$ $\boxed{\text{(e) is true.}}$

So. A is uncountable. ((a) is true.

From (b) $|A| = c^c \approx 2^c > c = |\mathbb{R}|$.

$\Rightarrow$ A is not similar to $\mathbb{R}$.

For c: $|\mathcal{P}(\mathbb{R})| = 2^c = c^c = |\mathbb{R}|$

$\therefore$ c is true.

For d: $|\mathcal{P}(\mathbb{N})| = 2^{\aleph_0} = c < 2^c = c^c = |A|$

$\therefore A \not\sim \mathcal{P}(\mathbb{N})$.

$\therefore$ d is not true.

(4) $A = \{ f : \mathbb{N} \to \mathbb{N} : f$ is function$\}$ then verify following statement →

(a) A is uncountable.

(b) A is similar to $\mathbb{R}$ (HT)

(c) A is similar to $\mathcal{P}(\mathbb{R})$.

(d) A is countable.

[illegible].