* Method of Solution of Homogeneous Linear ordinary Differential Equations of order n with Constant co-efficients :-

⇒ In this section, we study the solutions of linear homogeneous diff. eqn of higher order in which all of the co-efficients are real constants.

First we define the differential operator D. Here, D stands for $\frac{d}{dx}$, D^n for $\frac{d^n}{dx^n}$ and so on.

The index of D indicates the number of times the operation of differentiation must be carried out. For example, D^2x^2 implies that we must differentiate x^2 twice. Thus, $D^2x^2 = 2$

The operator D is a linear operator. The following results are valid for the operator D:

(i) $D(c_1f_1 + c_2f_2) = c_1Df_1 + c_2Df_2$ where f_1, f_2 are functions of x and c_1, c_2 are constants.

(ii) $D^m + D^n = D^n + D^m$

(iii) $D^mD^n = D^nD^m = D^{m+n}$

(iv) $(D-a)(D-b) = (D-b)(D-a)$ where a, b are constants,

If a function f is sufficiently differentiable function, then we can write

$$Df = f', \quad D^2f = D(Df) = D(f') = f'', \ldots \quad D^nf = f^{(n)}, \text{ etc.}$$

We define $D^0 = 1$ so that 1 is the operator defined by $1(f) = f$. Therefore, $D^0(f) = f$.

We now define the operator L by

$$L = P_0\frac{d^n}{dx^n} + P_1\frac{d^{n-1}}{dx^{n-1}} + \cdots + P_{n-1}\frac{d}{dx} + P_n$$

$$= P_0D^n + P_1D^{n-1} + \cdots + P_{n-1}D + P_n$$

$$= F(D) \text{ (say)}$$

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where $F(D)$ is a polynomial in D so that

$$Ly = P_0 D^n y + P_1 D^{n-1} y + \cdots + P_{n-1} D y + P_n y$$
$$= (P_0 D^n + P_1 D^{n-1} + \cdots + P_{n-1} D + P_n) y$$
$$= F(D) y$$

For example, the linear differential equation.

$$\frac{d^4y}{dx^4} + 4\frac{d^3y}{dx^3} - 5\frac{d^2y}{dx^2} - 36\frac{dy}{dx} - 36y = 0.$$

can be represented by

$$L(y) \equiv (D^4 + 4D^3 - 5D^2 - 36D - 36) y = 0$$

Now we are in a position to study the method of the solution of the homogeneous linear differential equation with real constant co-efficients.

Therefore, we shall be concerned with the equation.

$$P_0 \frac{d^ny}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + \cdots + P_{n-1}\frac{dy}{dx} + P_n y = 0$$

$$\text{or,} \quad P_0 D^n y + P_1 D^{n-1} y + \cdots + P_{n-1} Dy + P_n y = 0 \quad \cdots (1)$$

where $P_0 (\neq 0)$, $P_1, P_2, \cdots, P_{n-1}, P_n$ are real constants. We shall show that the general sol^n of (1) can be found explicitly.

A clue for the method of solving the eq^n (1) can be found by simple examining the equation. In other words, we want to find a function $y(x)$ which satisfies (1) for all x belonging to some interval I.

One such function that has been studied extensively in the elementary calculus is the exponential function e^{mx} (m being a constant, real or imaginary). So we shall try to find the solution of the equation (1) of the form $y(x) = e^{mx}$ for suitably chosen value of m.

Assuming that $y = e^{mx}$ is a sol^n of (1) for certain values of m, we have

$$Dy = me^{mx}, \; D^2y = m^2e^{mx}, \cdots, D^{n-1}y = m^{n-1}e^{mx}, \; D^ny = m^ne^{mx}$$

Thus from (1) we find that

$$P_0 m^n e^{mx} + P_1 m^{n-1} e^{mx} + \cdots + P_{n-1} m e^{mx} + P_n e^{mx} = 0$$

$$\Rightarrow e^{mx}\left(P_0 m^n + P_1 m^{n-1} + \cdots + P_{n-1} m + P_n \right) = 0$$

since $e^{mx} \neq 0$ for all $x \in I$, we obtain the polynomial equation in the unknown m as given by

$$P_0 m^n + P_1 m^{n-1} + \cdots + P_{n-1} m + P_n = 0 \quad \cdots (2)$$

The equation (2) is known as the auxillary or characteristic eqn of the diff. eq^n (1).

If $y = e^{mx}$ is a solution of (1) then we see that the constant m must satisfy the relation (2). Hence to solve the eq^n (1), we have to write the auxiliary eqn (2) and solve it for m,

The eq^n (2) is of nth degree and has n roots. According to the nature of the roots of (2), we consider the following cases.

Case I: Distinct real roots :-

Let $m_1, m_2, \cdots, m_n$ be n real and distinct roots of (2). Then $e^{m_1 x}, e^{m_2 x}, \cdots, e^{m_n x}$ are n distinct sol^n of (1).

$\therefore$ The general sol^n of (1) is

$$y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$$

where $c_1, c_2, \cdots, c_n$ are arbitrary constants.

Case II: Real multiple roots :-

Let m_1 be the roots of the eqn (2) of multiplicity r i,e $m = m_1$ is repeated r times

$\therefore$ The general sol^n of (1) is

$$y = (c_1 + c_2 x + \cdots + c_r x^{r-1}) e^{m_1 x} + c_{r+1} e^{m_{r+1} x} + \cdots + c_n e^{m_n x}$$

Case III :- Complex conjugate roots.

If the auxiliary eqn has a pair of complex roots, say $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$, then the corresponding part of the solution is

$$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$$

in which $A = C_1 + C_2$ and $B = i(C_1 - C_2)$ are arbitrary constants.

If the above pair of complex roots occurs twice in the auxiliary eqn, then the corresponding part of the solution is

$$y = e^{\alpha x}\{(A_1 + A_2 x)\cos\beta x + (B_1 + B_2 x)\sin\beta x\},$$

where A_1, A_2, B_1, B_2 being arbitrary constants.

Example:- Solve $(D^2+1)^3(D^2+D+1)^2 y = 0$

Soln :. The given eqn is

$$(D^2+1)^3(D^2+D+1)^2 y = 0 \quad ---(1)$$

Let $y = e^{mx}$ (m being a constant) be a soln of (1).

Then its auxiliary eqn is

$$(m^2+1)^3(m^2+m+1)^2 = 0$$

$$\Rightarrow m = \pm i, \pm i, \pm i, -\tfrac{1}{2} \pm i\tfrac{\sqrt{3}}{2}, -\tfrac{1}{2} \pm i\tfrac{\sqrt{3}}{2}$$

Hence the general soln of (1) is

$$y(x) = (C_1 + C_2 x + C_3 x^2)\cos x + (C_4 + C_5 x + C_6 x^2)\sin x$$
$$+ e^{-x/2}\left\{(C_7 + C_8 x)\cos\left(\tfrac{\sqrt{3}}{2}x\right) + (C_9 + C_{10} x)\sin\left(\tfrac{\sqrt{3}}{2}x\right)\right\}$$

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Home work! -

(1) Solve $l\frac{d^2\theta}{dt^2}+g\theta=0$

(2) Solve $\frac{d^2s}{dt^2}+\frac{g}{l}(s-l)=0$ (l, g, e being constants).

(3) Solve $\frac{d^4y}{dx^4}-n^4y=0$

Solution of Non-homogeneous Linear ordinary Differential Equations with constant co-efficients :-

⇒ In this section, we study the methods for finding the general solⁿ of non-homogeneous linear ordinary differential equations with constant co-efficients of the form

$$L(y) \equiv P_0 y^{(n)}(x) + P_1 y^{(n-1)}(x) + \cdots + P_{n-1} y'(x) + P_n y(x) = q(x) \quad \cdots (1)$$

where $P_0 (\neq 0), P_1, \cdots, P_{n-1}, P_n$ are real constants when the general solⁿ of the corresponding homogeneous eqⁿ $Ly=0$ is known,

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particular soln (a soln not containing any arbitrary constant) of the non-homogeneous eqn (1), then the general soln of the eqn (1) is given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p(x) \quad \text{--- (2)}$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

<u>Proof:</u> Since $y_p(x)$ is a particular soln of (1), we have

$$L[y_p(x)] \equiv P_0 y_p^{(n)}(x) + P_1 y_p^{(n-1)}(x) + \cdots + P_n y_p(x) = g(x) \quad \text{--- (3)}$$

subtracting (3) ~~to~~ from (2), we get

$$P_0\{y^n(x) - y_p^{(n)}(x)\} + P_1\{y^{(n-1)}(x) - y_p^{(n-1)}(x)\} + \cdots + P_{n-1}\{y'(x) - y_p'(x)\} + P_n\{y(x) - y_p(x)\} = 0$$

$$\Rightarrow P_0 Z^{(n)} + P_1 Z^{(n-1)}(x) + \cdots + P_{n-1} Z'(x) + P_n Z(x) = 0 \quad \text{--- (4)}$$

$$\Rightarrow L(Z) = 0,$$

where $Z(x) = y(x) - y_p(x)$

Eqn (4) is the corresponding homogeneous eqn of (3). Thus the general soln of (4) is

$$Z(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$$

$$\Rightarrow y(x) - y_p(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$$

$$\Rightarrow y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p(x) \quad \text{--- (5)}$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

Since the solⁿ (5) contains n arbitrary constants, it is the general solⁿ of (3). This completes the proof.

Ex: Solve $(D^2 - 3D + 2)y = e^{3x}$.

Solⁿ:- Let $y = e^{mx}$ (m being a constant) be a solⁿ of the corresponding homogeneous eqⁿ of

$$(D^2 - 3D + 2)y = e^{3x} \quad \text{---} \; (1)$$

Then the auxiliary eqⁿ of the corresponding homogeneous eqⁿ is

(3) $$m^2 - 3m + 2 = 0 \Rightarrow (m-2)(m-1) = 0 \Rightarrow m = 1, 2$$

Thus the complementary function of (1) is

$$y_c(x) = c_1 e^x + c_2 e^{2x}$$

where c_1, c_2 are arbitrary constants.

Now, the particular integral of (1) is

$$y_p(x) = [F(D)]^{-1}(e^{3x}), \text{ where } F(D) = D^2 - 3D + 2$$

$$= \frac{1}{F(3)} e^{3x} = \frac{1}{2} e^{3x}$$

Hence the required general solⁿ of (1) is

$$y = y_c(x) + y_p(x)$$

$$= c_1 e^x + c_2 e^{2x} + \frac{1}{2} e^{3x}$$

where c_1, c_2 are arbitrary constants.

Home work :-

(1) Solve $(D^3-6D^2+12D-8)y = 18e^{2x}$

(2) Solve $(D^2+1)y = \cos 2x$

(3) Solve $(D^3+1)y = \cos(2x+3)$

(4) Solve $(D^2+2D+1)y = x\cos x$

(5) Solve $(D^2-2D+1)y = xe^x \sin x$

(6) Solve $(D^2+1)y = \operatorname{cosec} x$.