Power Series Solⁿ of a Linear Diffⁿ Eqⁿ

Introduction:

Let us consider the diffⁿ eqⁿ

$$y'' + y = 0.$$

If we try to solve the above eqⁿ we can find that $y = \sin x$ & $y = \cos x$ are two linearly independent solⁿ of the above eqⁿ. But the situation w.r.to the eqⁿ

$$xy'' + y' + xy = 0$$

is quite different. For the eqⁿ can not be solved in terms of elementary functions (i.e., algebraic function, transcendental functions occuring in calculus i.e., trigonometric, exponential and logarithmic functions and all other that can be formed from these by adding, subtracting, multiplying, dividing or forming a function of functions).

Infact there is no known type of second order linear eqⁿ — apart from those with constant coeffⁿ and equations reducible to those by changes of independent variable — which can be solved in terms of elementary functions.

Basic concepts.

Consider the 2nd order homoⁿ diffⁿ eqⁿ (linear)

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0 \longrightarrow (1)$$

and suppose that the eqⁿ has no solution that can be expressed as a finite linear combination of known elementary fns. Let us assume that it does have a solution in infinite series form

$$c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-x_0)^n \longrightarrow (2)$$

$c_0, c_1, \ldots c_n$ are constants

An expression of this form is called power series in $(x-x_0)$.

Let us assume that the diffⁿ eqⁿ (1) has a power series solution of the form (2). Assuming that this assumption is valid, we can proceed to determine the coeffⁿs. $c_0, c_1, c_2 \ldots$ in such a manner that the expression satisfy the eqⁿ.

But under what condition is this assumption actually valid? Without knowing the answer it would be quite absurd to try to find a solution of the form (2). For that we first introduce some basic definition.

Let us consider the equivalent normalized form of eqⁿ (1).

i.e., $$y'' + P_1(x)y' + P_2(x)y = 0 \longrightarrow (3)$$

where $P_1(x) = \dfrac{a_1(x)}{a_0(x)}$ & $P_2(x) = \dfrac{a_2(x)}{a_0(x)}$.

Defⁿ A fn. f is said to be analytic at $x = x_0$ if its Taylor series about x_0,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

exists and converges to $f(x)$ for all x in some open interval including x_0.

For example, the rational function defined by $\dfrac{1}{x^2-3x+2}$ is analytic except at $x=1$ and $x=2$

Defⁿ The point x_0 is called an ordinary point of (1) if of the fns. P_1 and P_2 of (3) are analytic at x_0 either (or both) of these fns. is not analytic at then x_0 is called a singular point of eqⁿ (1).

Theorem.

Hypothesis : The point x_0 is an ordinary point of the diffⁿ eqⁿ ①

Conclusion : The diffⁿ eqⁿ ① has two non-trivial linearly independent power series solution of the form

$$\sum_{n=0}^{\infty} c_n (x-x_0)^n$$

and these power series converge in some interval $|x-x_0| < R$ (where $R>0$) about x_0.

This theorem gives us a sufficient condition for the existence of power series solution of the diffⁿ eqⁿ ① and ~~answer~~ answered our earlier raised question.

Method.

Now how do we determine the coefficients $c_0, c_1 \cdots$ in the expression (2) so that it satisfy eqⁿ ①.

Assuming x_0 is an ordinary point of eqⁿ ① let us denote solⁿ as $y = \sum_{n=0}^{\infty} c_n (x-x_0)^n \longrightarrow (4)$

Since the series (4) converges on an interval $|x-x_0|<R$ about x_0, it may be differentiated term by term on the interval twice to obtain

$$\frac{dy}{dx} = y' = \sum_{n=1}^{\infty} n c_n (x-x_0)^{n-1} \longrightarrow (5)$$

and $$\frac{d^2y}{dx^2} = y'' = \sum_{n=2}^{\infty} n(n-1) c_n (x-x_0)^{n-2} \longrightarrow (6)$$

Substituting (5) and (6) in eqⁿ ① and simplifying the resulting expression so that it takes the form

$$K_0 + K_1 (x-x_0) + K_2 (x-x_0)^2 + \cdots = 0 \longrightarrow (7)$$

where the coeffⁿs $K_i\ (i=0,1,2\cdots)$ are fns. of certain coefficients c_n. In order that (7) be valid for all x in $|x-x_0|<R$, we must set

$$K_0 = K_1 = K_2 = \cdots = 0.$$

This leads to a set of condition that must be satisfied by the various coefficients c_n.

If the c_n are choosen to satisfy the set of conditions, then the resulting series (4) is the desired solution of the diffⁿ eqⁿ ①.

Problem Find the power series solution of the differential eqⁿ

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2+2)y = 0$$

in power of x (i.e, about $x_0=0$).

Solⁿ: Observe that $x_0=0$ is an ordinary point of the given diffⁿ eqⁿ (Here $P_1(x) = x$, $P_2(x) = (x^2+2)$).

Hence two linearly independent solⁿ of the desired type exists.

We assume a solution of the form

$$y = \sum_{n=0}^{\infty} c_n x^n$$

Differentiating term by term

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

Substituting in the given diffⁿ eqⁿ we get,

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x\sum_{n=1}^{\infty} n c_n x^{n-1} + x^2\sum_{n=0}^{\infty} c_n x^n + 2\sum_{n=0}^{\infty} c_n x^n = 0.$$

i.e, $$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^{n+2} + 2\sum_{n=0}^{\infty} c_n x^n = 0$$

We shall rewrite the first and third summation in such a way that x in each of these summation will have the same exponents n.

Thus the eqⁿ can be rewritten as

$$\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n + \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=2}^{\infty} c_{n-2} x^n + 2\sum_{n=0}^{\infty} c_n x^n = 0$$

Now writing individually the terms in each summation that do not belong to the common range 2 to ∞ we get

$$\left\{2c_2 + 6c_3 x + \sum_{n=2}^{\infty}(n+2)(n+1)c_{n+2}x^n\right\} + \left\{c_1 x + \sum_{n=2}^{\infty} n c_n x^n\right\}$$
$$+ \sum_{n=2}^{\infty} c_{n-2}x^n + \left\{2c_0 + 2c_1 x + 2\sum_{n=2}^{\infty} c_n x^n\right\} = 0.$$

or, $(2c_0+2c_2)+(3c_1+6c_3)x+\sum_{n=2}^{\infty}[(n+2)(n+1)c_{n+2}+(n+2)c_n+c_{n-2}]x^n=0$

~~this eq~~ The above eq^n is to be valid for all x in the interval of convergence $|x-x_0|<R$, the coefficients of each power of x in the left member must be equated to zero

i.e, $2c_0+2c_2=0 \Rightarrow c_0=-c_2$

and $3c_1+6c_3=0 \Rightarrow c_3=-\frac{1}{2}c_1$

$(n+2)(n+1)c_{n+2}+(n+2)c_n+c_{n-2}=0,\ n\geq 2$

i.e, $c_{n+2}=-\dfrac{(n+2)c_n+c_{n-2}}{(n+2)(n+1)},\ n\geq 2$

$\therefore\ c_4=-\dfrac{4c_2+c_0}{12}$ (when $n=2$)

$=\frac{1}{4}c_0$ $(\because c_2=-c_0)$

$c_5=-\dfrac{5c_3+c_1}{20}$

$=\frac{1}{40}c_1$ and so on.

Substituting all c_i's we get the sol^n as

$y=c_0+c_1x+c_2x^2+c_3x^3+c_4x^4+\cdots$

$=c_0+c_1x-c_0x^2-\frac{1}{2}c_1x^3+\frac{1}{4}c_0x^4+\frac{1}{40}c_1x^5+\cdots$

$=c_0(1-x^2+\frac{1}{4}x^4+\cdots)+c_1(x-\frac{1}{2}x^3+\frac{1}{40}x^5+\cdots)$.

Ex (i) Find power series sol^n in powers of x

i) $\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}+y=0$.

ii) $(x^2+1)\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}+xy=0$.

iii) $\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}-y=0,\ y(0)=1,\ y'(0)=0$.