:- Method of Variation of Parameter :-

We recall the Second Order linear differential equation with Constant Co-efficients given in (1)

$$\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2 y = \phi(x) \quad \text{---} (1)$$

Where P_1 and P_2 are constants and $\phi(x)$ is a function of x only.

Below we describe two steps to solve the differential equation (1) by Variation of Parameter technique :-

Step-1 : First find two linearly independent solutions y_1 and y_2 for the reduced form of the diff. eq^n (1)

Then the general Solution of the Corresponding reduced eqn (C.F) is given by $y_c = c_1 y_1 + c_2 y_2$, where c_1 and c_2 are two arbitrary Constants. Also the linear independency of the Solutions y_1 and y_2 Can be checked by finding Wronskian of them.

Step 2: Let the Complete Soln or the General Soln of the diff. eqn (1) be

$$y(x) = u(x) y_1(x) + v(x) y_2(x) \quad \text{---} \quad (2)$$

Which is formed by replacing two arbitrary Constants c_1 and c_2 by two functions $u(x)$ and $v(x)$ which are to be determined in the following steps.

Step 3:— Differentiating (2) w.r.to x we have,

$$\frac{dy}{dx} = \left(u(x)\frac{dy_1}{dx} + v(x)\frac{dy_2}{dx}\right) + \left(y_1(x)\frac{du}{dx} + y_2(x)\frac{dv}{dx}\right) \quad \text{--- (3)}$$

The function $u(x)$ and $v(x)$ are to be chosen in such a way that

$$y_1(x)\frac{du}{dx} + y_2(x)\frac{dv}{dx} = 0 \quad \text{--- (4)}$$

Then (3) reduces to $\frac{dy}{dx} = u(x)\frac{dy_1}{dx} + v(x)\frac{dy_2}{dx}$ --- (5)

Step 4: Again differentiating (5) w.r.to x we have,

$$\frac{d^2y}{dx^2} = u(x)\frac{d^2y_1}{dx^2} + v(x)\frac{d^2y_2}{dx^2} + \left(\frac{dy_1}{dx}\cdot\frac{du}{dx} + \frac{dy_2}{dx}\cdot\frac{dv}{dx}\right) \quad \text{--- (6)}$$

Now substituting the values of $\frac{dy}{dx}$ (from (5)) and $\frac{d^2y}{dx^2}$ (from (6)) in the diff. eq^n (1) and simplifying

We have $\frac{dy_1}{dx}\cdot\frac{du}{dx}+\frac{dy_2}{dx}\cdot\frac{dv}{dx}=\phi(x)$ (7)

Step-5: Since $y_1(x)$, $y_2(x)$ and $\phi(x)$ are known functions of x only therefore from (5) and (7) we can determine the unique values of $\frac{du}{dx}$ and $\frac{dv}{dx}$.

Clearly both of them must be functions of x only. Let us assume

$$\frac{du}{dx}=F_1(x) \text{ and } \frac{dv}{dx}=F_2(x)$$

Step-6:- On integration we have,

$$u(x)=\int F_1(x)\,dx+A \text{ and}$$

$$v(x)=\int F_2(x)\,dx+B,$$ where A & B are two arbitrary constants.

Step-7: Substituting the values of $u(x)$ and $v(x)$ in (2) we have the required general soln which is as follows:

$$y = Ay_1(x) + By_2(x) + y_1(x)\int F_1(x)\,dx + y_2(x)\int F_2(x)\,dx \quad \text{---} (8)$$

where A and B are arbitrary constant and $F_1(x)$, $F_2(x)$ are the soln of the diff. eqn $\frac{du}{dx} = F_1(x)$ and $\frac{dv}{dx} = F_2(x)$ respectively.

~~Here~~ the above discussed steps (Step 1 - Step 7) are required to solve a second order linear diff. eqn by variation of parameters technique and the whole method to determine the general soln is known as variation of parameters.

Alternative Method:—

If y_1 and y_2 be two independent Soln of the eqn (2) then the general Soln of the eqn (1) will be

$y = y_c + y_p$ where $y_c = C.F$ of (1)

$= Ay_1 + By_2$ and $y_p = P.I$ of (1)

$= uy_1 + vy_2$

where A, B are two arbitrary constants and u, v can be determined from the following formula:

$$u = -\int \frac{y_2 \phi(x)}{W} dx \text{ and } v = \int \frac{y_1 \phi(x)}{W} dx ;$$

W being the Wronskian of y_1 & y_2

Example:

Consider the diff. eqn

$$\frac{d^2y}{dx^2} + a^2 y = \sin ax \quad \text{---} (1)$$

Let $y = e^{mx}$ be a trial soln of the reduced form of (1)

Then the auxiliary eqn is $m^2 + a^2 = 0$

giving $m = \pm ai$

Therefore, $C.F = A\cos ax + B\sin ax$, where A & B are arbitrary constants.

Let $y_1 = \cos ax$, $y_2 = \sin ax$

Now $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a \neq 0$

Thus the functions y_1 and y_2 are linearly independent.

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	10	11	12	13	14	15	16
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Let the P.I be $y_p = u y_1 + v y_2$

$$= u \cos ax + v \sin ax$$

Therefore $u = -\int \frac{y_2 \phi(x)}{W} dx$ & $v = \int \frac{y_1 \phi(x)}{W} dx$.

Where $\phi(x) = \sin ax$

Therefore $u = -\int \frac{\sin ax \sin ax}{a} dx$

$$= -\frac{1}{2a}\int (1-\cos 2ax) dx = -\frac{1}{2a}\left(x - \frac{\sin 2ax}{2a}\right)$$

$$v = \int \frac{\cos ax \sin ax}{a} dx = \frac{1}{2a}\int \sin 2ax\, dx = -\frac{1}{4a^2}\cos 2ax$$

Therefore $P.I = -\frac{1}{2a}\left(x\cos ax - \frac{\sin 2ax \cos ax}{2a}\right) - \frac{1}{4a^2}\cos 2ax$

$$= -\frac{1}{2a} x \cos ax + \frac{1}{4a^2} \sin ax$$

Therefore the general sol^n is

$$y = C.F + P.I$$

$= A\cos ax + B\sin ax - \frac{1}{2a} x\cos ax + \frac{1}{4a^2}\sin ax.$

$= A\cos ax + B_1 \sin ax - \frac{1}{2a} x \cos ax$

where $B_1 = \left(B + \frac{1}{4a^2}\right)$

and A are two arbitrary constants.

Home work:

1) Solve $\frac{d^2y}{dx^2} + a^2y = \sec ax$ by method of variation of Parameter.

2) Solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \frac{e^x}{1+e^x}$

3) Solve. $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$.

1	2	3	4	5	6	7
8	9	10	11	12	13	14
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22	23	24	25	26	27	28
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