Case iii $c = b$

The proof is similar as that of case (ii).

Fundamental theorem of Integral calculus

If (i) $\int_a^b f(x)dx$ exists ; and

(ii) $\exists$ a function $\phi(x)$ such that
$$\phi'(x) = f(x) \text{ on } [a,b]$$

then $\int_a^b f(x)dx = \phi(b) - \phi(a)$

proof: Let $P = \{a = x_0, x_1, \ldots, x_{r-1}, x_r, \ldots, x_n = b\}$ be a partition of $[a,b]$ with component intervals $I_r = [x_{r-1}, x_r], r = 1,2,\ldots,n$. Let
$$M_r = \sup_{x \in I_r} f(x) \text{ and } m_r = \inf_{x \in I_r} f(x) \text{ for } r = 1,2,\ldots,n.$$

Since $\phi'(x) = f(x)$ for all $x \in [a,b]$. Therefore $\phi(x)$ satisfies all the condition of Lagranges Mean value theorem on each $I_r = [x_{r-1}, x_r]$. Thus $\exists$ a point $\xi_r \in (x_{r-1}, x_r)$ such that
$$\phi(x_r) - \phi(x_{r-1}) = (x_r - x_{r-1})\phi'(\xi_r)$$
$$= (x_r - x_{r-1})f(\xi_r) \text{ for } r = 1,2,\ldots,n$$

So, $\sum_{r=1}^{n}(x_r - x_{r-1})f(\xi_r) = \sum_{r=1}^{n}\{\phi(x_r) - \phi(x_{r-1})\}$
$$= \phi(b) - \phi(a)$$

Now $m_r \le f(\xi_r) \le M_r$ for $r = 1,2,\ldots,n$.

i.e. $\sum_{r=1}^{n} m_r(x_r - x_{r-1}) \le \sum_{r=1}^{n} f(\xi_r)(x_r - x_{r-1}) \le \sum_{r=1}^{n}(x_r - x_{r-1})M_r$,

i.e, $L(P;f) \le \phi(b) - \phi(a) \le U(P;f)$

Since P is an arbitrary partition of $[a,b]$, therefore the set of all lower sums $\{L(P;f) : P \text{ is a partition of } [a,b]\}$ is bounded above by $\phi(b) - \phi(a)$.

So $\int_a^b f(x)dx \le \phi(b) - \phi(a)$ - - - (1)

Again the set of all upper sums $\{U(P;f) : P \text{ is a partition of } [a,b]\}$ is bounded below by $\phi(b) - \phi(a)$.

So, $\overline{\int_a^b} f(x)\,dx \geq \phi(b)-\phi(a) \quad \ldots (2)$

Thus from (1) & (2)

$$\underline{\int_a^b} f(x)\,dx \leq \phi(b)-\phi(a) \leq \overline{\int_a^b} f(x)\,dx.$$

Since $f(x)$ is integrable on $[a,b]$,

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx \text{ and so } \int_a^b f(x)\,dx = \phi(b)-\phi(a)$$

This completes the proof of the theorem.

<u>Theorem:</u> Let $f(x)$ and $\phi(x)$ be bounded integrable on $[a,b]$ and $\phi(x)$ keep the same sign on $[a,b]$.

Then $\displaystyle\int_a^b f(x)\phi(x)\,dx = \mu\int_a^b \phi(x)\,dx$,

where $m \leq \mu \leq M$, m and M being the greatest lower and least upper bound of $f(x)$ on $[a,b]$.

<u>proof:</u> ~~First~~ suppose that $\phi(x)$ is non-negative on $a \leq x \leq b$.

Since $m \leq f(x) \leq M$ for $a \leq x \leq b$, therefore

$$m\phi(x) \leq f(x)\phi(x) \leq M\phi(x) \text{ for } a \leq x \leq b.$$

Now, $m\phi(x), f(x)\phi(x), M\phi(x)$ are all integrable functions on $[a,b]$ and so $\displaystyle\int_a^b m\phi(x)\,dx \leq \int_a^b \phi(x)f(x)\,dx \leq \int_a^b M\phi(x)\,dx$

$$\text{i.e. } m\int_a^b \phi(x)\,dx \leq \int_a^b f(x)\phi(x)\,dx \leq M\int_a^b \phi(x)\,dx$$

Therefore $\exists$ a real number μ satisfying $m \leq \mu \leq M$ such that

$$\int_a^b f(x)\phi(x)\,dx = \mu\int_a^b \phi(x)\,dx$$

If ~~f~~ $\phi(x)$ be negative throughout $[a,b]$. The proof is similar.

<u>Corollary:</u> If in addition $f(x)$ be continuous on $[a,b]$ then

$$\int_a^b f(x)\phi(x)\,dx = f(\xi)\int_a^b \phi(x)\,dx \quad \text{for some } \xi \in [a,b].$$

<u>proof:</u> Since $m \leq \mu \leq M$ and $f(x)$ is continuous on $[a,b]$ therefore, ~~∃ a real nu~~ by Intermediate value property $\exists$ a point $\xi \in [a,b]$ such that $\mu = f(\xi)$

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Hence from the above theorem

$$\int_a^b f(x)\phi(x)dx = f(\xi)\int_a^b \phi(x)dx, \quad a \le \xi \le b.$$

NOTE: The above theorem known as first mean value theorem for integrals.

• Abels Inequality

If (i) $a_1, a_2, \ldots, a_n$ is a non-increasing sequence of positive real numbers

(ii) $v_1, v_2, \ldots, v_n$ is any set of n-real numbers, and

(iii) h, H be two real numbers, such that

$$h < v_1 + v_2 + \cdots + v_p < H \text{ for } 1 \le p \le n$$

then $\quad a_1 h < a_1 v_1 + a_2 v_2 + \cdots + a_n v_n < a_1 H$

proof: Let $s_p = v_1 + v_2 + \cdots + v_p, \; 1 \le p \le n$ then

$$\sum_{r=1}^{n} a_r v_r = a_1 v_1 + a_2 v_2 + a_3 v_3 + \cdots + a_n v_n$$

$$= a_1 s_1 + a_2 (s_2 - s_1) + a_3 (s_3 - s_2) + \cdots + a_n (s_n - s_{n-1})$$

$$= (a_1 - a_2) s_1 + (a_2 - a_3) s_2 + (a_3 - a_2) s_3 + \cdots + (a_{n-1} - a_n) s_{n-1} + a_n s_n$$

Since by (i), $a_1, a_2, \ldots, a_n$ is a non-increasing sequence of positive real numbers,

$a_1 - a_2, a_2 - a_3, \ldots, a_{n-1} - a_n$ are all non-negative real numbers

Therefore by (iii)

$$\sum_{r=1}^{n} a_r v_r > (a_1 - a_2) h + (a_2 - a_3) h + \cdots + (a_{n-1} - a_n) h + a_n h$$

$$= a_1 h \text{ and}$$

$$< (a_1 - a_2) H + (a_2 - a_3) H + \cdots + (a_{n-1} - a_n) H + a_n H$$

$$= a_1 H$$

Thus $a_1 h < \sum_{r=1}^{n} a_r v_r < a_1 H$.

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$a_1 h < a_1 v_1 + \cdots + a_n v_n < a_1 H.$

now, we shall state & prove 2nd mean value theorem

Statement (Bonets form): Let $f(x)$ be bounded monotone non-increasing and never negative on $[a,b]$ and let $\phi(x)$ be bounded integrable on $[a,b]$. Then $\exists$ a value of x, say ξ on $[a,b]$ such that

$$\int_a^b f(x)\,\phi(x)\,dx = f(a)\int_a^{\xi}\phi(x)\,dx.$$

~~proof~~

statement (weierstrass form): Let $f(x)$ be bounded monotonic on $[a,b]$ and let $\phi(x)$ be bounded integrable on $[a,b]$. Then there exist at least one value of x, say ξ in $[a,b]$ such that

$$\int_a^b f(x)\,\phi(x)\,dx = f(a)\int_a^{\xi}\phi(x)\,dx + f(b)\int_{\xi}^{b}\phi(x)\,dx.$$

proof: let $P = \{a = x_0, x_1, \ldots, x_{r-1}, x_r, \ldots, x_{n-1}, x_n = b\}$ be a partition of $[a,b]$ with component intervals $I_r = [x_{r-1}, x_r]$ $r = 1,2,\ldots,n$. Let δ_r = length of $I_r = x_r - x_{r-1}$ $(r = 1,2,\ldots n)$.

Let M_r, m_r be the respective supremum and infimum of $\phi(x)$ on ~~[a,b]~~ I_r $(r=1,2,\ldots n)$. Let $\xi_1 = a$ and ξ_r $(r \neq 1)$ be any arbitrary point of I_r. Then

$$m_r\delta_r \leq \int_{x_{r-1}}^{x_r}\phi(x)\,dx \leq M_r\delta_r \quad \text{and} \quad m_r\delta_r \leq \phi(\xi_r)\,\delta_r \leq M_r\delta_r$$

putting $r = 1,2,\ldots,p$ $(1 \leq p \leq n)$ and then adding, we get

$$\sum_{r=1}^{p} m_r\delta_r \leq \int_a^{x_p}\phi(x)\,dx \leq \sum_{r=1}^{p} M_r\delta_r \quad \text{and} \quad \sum_{r=1}^{p} m_r\delta_r \leq \sum_{r=1}^{p}\phi(\xi_r)\,\delta_r \leq \sum_{r=1}^{p} M_r\delta_r$$

Therefore

$$\left|\sum_{r=1}^{p}\phi(\xi_r)\,\delta_r - \int_a^{x_p}\phi(x)\,dx\right| \leq \sum_{r=1}^{p}(M_r - m_r)\,\delta_r \leq \sum_{r=1}^{n}(M_r - m_r)\,\delta_r$$

i.e., $$\int_a^{x_p}\phi(x)\,dx - \sum_{r=1}^{n}(M_r - m_r)\,\delta_r \leq \sum_{r=1}^{p}\phi(\xi_r)\,\delta_r \leq \int_a^{x_p}\phi(x)\,dx + \sum_{r=1}^{n}(M_r - m_r)\,\delta_r$$

Since $\int_a^x \phi(t)\,dt$ is a continuous function of x on $[a,b]$, it is bounded on $[a,b]$ and attains its bounds on $[a,b]$. Let M, m be the respective supremum and infimum of

$\int_a^x \phi(t)dt$ on $[a,b]$. Then

$$m - \sum_{r=1}^{n} (M_r - m_r)\delta_r \le \sum_{r=1}^{p} \phi(\xi_r)\delta_r \le M + \sum_{r=1}^{n} (M_r - m_r)\delta_r.$$

Now, we shall apply Abel's inequality.

Let $a_r = f(\xi_r)$, $v_r = \phi(\xi_r)\delta_r$, $h = m - \sum_{r=1}^{n}(M_r - m_r)\delta_r$ and

$H = M + \sum_{r=1}^{n} (M_r - m_r)\delta_r$. Then by Abel's inequality we have

$$a_1 h \le \sum_{r=1}^{n} f(\xi_r)\phi(\xi_r)\delta_r \le a_1 H, \text{ i.e.;}$$

$$f(a)\left\{m - \sum_{r=1}^{n}(M_r - m_r)\delta_r\right\} \le \sum_{r=1}^{n} f(\xi_r)\phi(\xi_r)\delta_r \le f(a)\left\{M + \sum_{r=1}^{n}(M_r - m_r)\delta_r\right\} \quad \text{---(1)}$$

Let now norm of $P \to 0$ so that

$$\sum_{r=1}^{n}(M_r - m_r)\delta_r \to 0 \text{ and } \sum_{r=1}^{n} f(\xi_r)\phi(\xi_r)\delta_r \to \int_a^b f(x)\phi(x)dx.$$

Hence from (1), letting $\|P\| \to 0$, we have

$$m f(a) \le \int_a^b f(x)\phi(x)\,dx \le M f(a)$$

So, $\exists$ a real number μ between m & M such that

$$\int_a^b f(x)\phi(x)\,dx = \mu f(a), \; m \le \mu \le M \quad \text{---(2)}$$

But μ is an intermediate value between the upper and lower bounds of the continuous function $\int_a^x \phi(t)dt$ on $[a,b]$ and so by the intermediate value property $\exists$ a point $\xi \in [a,b]$ such that

$$\mu = \int_a^{\xi} \phi(x)dx$$

So from (2)

$$\int_a^b f(x)\phi(x)\,dx = f(a)\int_a^{\xi}\phi(x)dx, \quad a \le \xi \le b.$$

This proves 2nd mean value theorem in Bonet's form.

<u>Weierstrass form:-</u>

First suppose that $f(x)$ be bounded monotone non-increasing on $[a,b]$. Then $f(x) - f(b)$ is bounded monotone non-increasing and never negative on $[a,b]$ and so by Bonet's form of 2nd mean value theorem, $\exists$ at least one value of x, say $\xi \in [a,b]$ such that

$$\int_a^{\xi}\{f(a)-f(b)\}\phi(x)dx = \{f(a)-f(b)\}\int_a^{\xi}\phi(x)dx$$

or, $$\int_a^b f(x)\phi(x)dx = f(a)\int_a^{\xi}\phi(x)dx + f(b)\int_{\xi}^b \phi(x)dx$$

Let us now, suppose that $f(x)$ be bounded monotone non-decreasing on $[a,b]$. Then $f(b)-f(x)$ is bounded monotone non-increasing and never negative on $[a,b]$. Hence again, by Bonet's form of 2nd Mean value theorem $\exists$ at least one value of x, say $\xi \in [a,b]$ such that

$$\int_a^{\xi}\{f(b)-f(x)\}\phi(x)dx = \{f(b)-f(a)\}\int_a^{\xi}\phi(x)dx$$

i.e., $$\int_a^b f(x)\phi(x)dx = f(a)\int_a^{\xi}\phi(x)dx + f(b)\int_{\xi}^b \phi(x)dx, \quad a\le\xi\le b.$$

The proof is now complete.

Problem: prove that $\left|\int_{x'}^{x''}\frac{\sin x}{x}dx\right| \le \frac{2}{x'}$ for $x''>x'>0$.

Solution: Let $f(x)=\frac{1}{x}$, $\phi(x)=\sin x$, $x\in[x',x'']$. Then $f(x)$ is bounded monotone non-increasing and never negative on $[x',x'']$. Also $\phi(x)$ is bounded and integrable on $[x',x'']$. So by Bonet's form of 2nd Mean value theorem $\exists$ at least one value of x say $\xi\in[x',x'']$ such that

$$\int_{x'}^{x''}\frac{\sin x}{x}dx = \frac{1}{x'}\int_{x'}^{\xi}\sin x\, dx \text{ ------ (i)}$$

$$\therefore \left|\int_{x'}^{x''}\frac{\sin x}{x}dx\right| = \frac{1}{x'}\left|\int_{x'}^{\xi}\sin x\,dx\right|$$

$$= \frac{1}{x'}\left|[-\cos x]_{x'}^{\xi}\right|$$

$$= \frac{1}{x'}\left|-\cos\xi+\cos x'\right|$$

$$\le \frac{1}{x'}\{|\cos\xi|+|\cos x'|\}$$

$$\le \frac{2}{x'}$$

$$\therefore \left|\int_{x'}^{x''}\frac{\sin x}{x}dx\right| \le \frac{2}{x'} \qquad x''>x'>0.$$

Problem: prove that $\left|\int_{x'}^{x''}\frac{\sin x}{x}dx\right| < \frac{4}{x'}$, for $x''>x'>0$.