Interchanging of limit and Integral:

Th: Let $\{f_n\}$ be a sequence of functions in $\mathcal{R}[a,b]$ and suppose that $\{f_n\}$ converges uniformly on $[a,b]$ to f. Then

$$f \in \mathcal{R}[a,b] \text{ and } \lim_{n\to\infty} \int_a^b f_n(x) = \int_a^b f(x)\,dx.$$

Proof: It follows from the cauchy criterion that given $\varepsilon > 0$ $\exists$ $H(\varepsilon)$ s.t if $m > n \geq H(\varepsilon)$ then

$$-\varepsilon \leq f_m(x) - f_n(x) \leq \varepsilon, \ \forall x \in [a,b].$$

$$\Rightarrow -\varepsilon(b-a) \leq \int_a^b f_m - \int_a^b f_n \leq \varepsilon(b-a).$$

Since $\varepsilon > 0$ is arbitrary, the sequence $\{\int_a^b f_n\}$ is cauchy seqn in $\mathbb{R}$ and therefore converges to some number, say $A \in \mathbb{R}$.

We now show $f \in \mathcal{R}[a,b]$ with integral A. If $\varepsilon > 0$ is given, let $K(\varepsilon)$ be such that if $m > K(\varepsilon)$ then $|f_m(x) - f(x)| < \varepsilon$ $\forall x \in [a,b]$. If $\dot{P} := \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ is any tagged partition of $[a,b]$ and if $m > K(\varepsilon)$ then

$$|S(f_m;\dot{P}) - S(f;\dot{P})| = \left|\sum_{i=1}^n \{f_m(t_i) - f(t_i)\}(x_i - x_{i-1})\right|$$

$$\leq \sum_{i=1}^n |f_m(t_i) - f(t_i)|(x_i - x_{i-1}).$$

$$\leq \sum_{i=1}^n \varepsilon(x_i - x_{i-1}) = \varepsilon(b-a).$$

We now choose $r \geq K(\varepsilon)$ s.t $|\int_a^b f_r - A| < \varepsilon$ and we let $\delta_{r,\varepsilon} > 0$ be such that $|\int_a^b f_r - S(f_r;\dot{P})| < \varepsilon$ whenever $\|\dot{P}\| < \delta_{r,\varepsilon}$. Then we have

$$|S(f;\dot{P}) - A| \leq |S(f;\dot{P}) - S(f_r;\dot{P})| + \left|S(f_r;\dot{P}) - \int_a^b f_r\right| + \left|\int_a^b f_r - A\right|$$

$$\leq \varepsilon(b-a) + \varepsilon + \varepsilon = \varepsilon(b-a+2).$$

But since $\varepsilon > 0$ is arbitrary, it follows that $f \in \mathcal{R}[a,b]$ and $\int_a^b f = A$.

Th: (Bounded convergence Theorem) Let $\{f_n\}$ be a sequence in $\mathcal{R}[a,b]$ that converges on $[a,b]$ to a function $f \in \mathcal{R}[a,b]$. Suppose also that there exists $B > 0$ such that $|f_n(x)| \leq B$ $\forall$ $x \in [a,b]$, $n \in \mathbb{N}$. then

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$$

(H.T).

Th: Dini's Theorem: (Statement only).

Suppose that (f_n) is a monotone seq^n of continuous functions on $I := [a,b]$ that converges on I to a continuous function f. then the convergence of the sequence is uniform.

problem:

V.V.V

① Suppose $\{f_n\}$ is a seq^n of continuous functions on an interval I that converges uniformly on I to a function f. If $\{x_n\} \subseteq I$ converges to $x_0 \in I$, show that $\lim_{n\to\infty} f_n(x_n) = f(x_0)$.

Sol^n: Since $\{f_n\}$ converges uniformly to f on I, for any $\epsilon > 0$ $\exists$ $K_1 \in \mathbb{N}$ s.t $n \geq K_1 \Rightarrow |f_n(x) - f(x)| < \epsilon/2$ $\forall x \in I$.

Since $\{x_n\} \in I$ so $n \geq K_1 \Rightarrow |f_n(x_n) - f(x_n)| < \epsilon/2$

Since each f_n is continuous on I and $\{f_n\}$ converges to f uniformly on I, we conclude that f is continuous on I.

$\therefore x_n \to x_0 \Rightarrow f(x_n) \to f(x_0)$.

Since $f(x_n) \to f(x_0)$ so for $\epsilon > 0$ $\exists$ $K_2 \in \mathbb{N}$ s.t

$$|f(x_n) - f(x_0)| < \epsilon/2 \quad \forall n \geq K_2$$

Now let $K = \max\{K_1, K_2\}$. Then

$$|f_n(x_n) - f(x_0)| \leq |f_n(x_n) - f(x_n) + f(x_n) - f(x_0)|$$

$$\leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x_0)|$$

$$\leq \epsilon/2 + \epsilon/2 = \epsilon \quad \forall n \geq K.$$

$$\therefore \boxed{\lim_{n\to\infty} f_n(x_n) = f(x_0)}$$

V.V.V

② Let $f: \mathbb{R} \to \mathbb{R}$ be uniformly continuous on $\mathbb{R}$ and let $f_n(x) = f(x + \frac{1}{n})$ for $x \in \mathbb{R}$. Show that $\{f_n\}$ converges uniformly on $\mathbb{R}$ to f.

Sol^n: Let $\epsilon > 0$. f is uniformly continuous on $\mathbb{R}$ so $\exists$ $\delta > 0$ s.t $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$.

Since $|x + \frac{1}{n} - x| = \frac{1}{n}$ and $\lim_{n\to\infty} \frac{1}{n} = 0$ so $\exists$ $K \in \mathbb{N}$ s.t

$$|\tfrac{1}{n}| \leq \delta, \quad \forall n \geq K$$

$\therefore$ for $n \geq K$

$\left|f(x+\frac{1}{n}) - f(x)\right| < \varepsilon$ whenever $|x-\frac{1}{n}-x| < \delta$.

$\Rightarrow \{f_n\}$ uniformly converges to f on $\mathbb{R}$.

(3) Let $g_n(x) = nx(1-x)^n$ for $x \in [0,1]$, $n \in \mathbb{N}$. Discuss the convergence of $\{g_n\}$ and $\{\int_0^1 g_n \, dx\}$.

(4) Show that $\lim \int_1^2 e^{-nx^2} dx = 0$.

(5) If $a > 0$, show that $\lim \int_a^\pi \frac{\sin nx}{nx} dx = 0$.
What happens if $a = 0$?

H.T.

Sem- IV

It is often useful to know whether the limit of a sequence of function is a continuous function, a differentiable function, or a Remann integrable function. unfortunately it is not always the case that the limit of a sequence of function prosses these useful properties.

Now we ~~do~~ study the properties of limit function. We start with a example —

(i) Let $g_n(x) = x^n$ for $x \in [0,1]$ $\forall n \in \mathbb{N}$.

Then we know $g_n \longrightarrow g(x) = \begin{cases} 0 & \text{for } 0 \le x < 1 \\ 1 & \text{for } x = 1 \end{cases}$

So $g(x) = \begin{cases} 0 & \text{for } 0 \le x < 1 \\ 1 & \text{for } x = 1 \end{cases}$ is limit function.

~~There is a question that is if~~

Clearly $\forall n$, g_n are continuous at $x=1$ but g is not continuous at $x = 1$.

Also $\{g_n\}$ does not converges uniformly to g on $[0,1]$.

(ii) Again each $g_n(x) = x^n$ has continuous derivative on $[0,1]$ However, the limit function g does not have a derivative at $x=1$, (why?).

(ii) Let $f_n(x) = \begin{cases} n^2 x & \text{for } 0 \le x \le 1/n \\ -n^2 (x - \frac{2}{n}) & \text{for } 1/n \le x \le 2/n \\ 0 & \text{for } 2/n \le x \le 1 \end{cases}$

Clearly $\forall n \in \mathbb{N}$, f_n is continuous on $[0,1]$. So $\forall n \in \mathbb{N}$ f_n is R-integrable.

$$\therefore \int_0^1 f_n(x)\,dx = \int_0^{1/n} f_n(x)\,dx + \int_{1/n}^{2/n} f_n(x)\,dx + \int_{2/n}^1 f_n(x)\,dx$$

$$= \int_0^{1/n} n^2 x\,dx + \int_{1/n}^{2/n} -n^2\left(x - \frac{2}{n}\right) + 0$$

$$= n^2 \left[\frac{x^2}{2}\right]_0^{1/n} - n^2 \left[\frac{x^2}{2} - \frac{2}{n}x\right]_{1/n}^{2/n}$$

$$= \frac{1}{2} - n^2 \left[\frac{4}{2n^2} - \frac{4}{2n^2} - \frac{1}{2n^2} + \frac{2}{n^2}\right]$$

$$= \frac{1}{2} + \frac{1}{2} - 2 = 1 - 2 = -1$$

Also $\lim_{n\to\infty} f_n(x) = 0$ $\forall x \in [0,1]$. Then $\int_0^1 f(x)\,dx = 0$.

$$\therefore \int_0^1 f(x)\,dx = 0 \neq (-1) = \lim_{n\to\infty} \int_0^1 f_n(x)\,dx$$

(iii) Let $h_n(x) := 2nxe^{-nx^2}$ $\forall x \in [0,1]$, $n \in \mathbb{N}$. check that $\int_0^1 h(x)\,dx \neq \lim_{n\to\infty} \int_0^1 h_n(x)\,dx$ (H.T).

Interchanging of limit and Integral

Th: Let $\{f_n\}$ be a sequence of functions in $\mathcal{R}[a,b]$ and suppose that $\{f_n\}$ converges uniformly on $[a,b]$ to f. Then $f \in \mathcal{R}[a,b]$ and $\lim_{n\to\infty} \int_a^b f_n(x) = \int_a^b f(x)\,dx$.

Proof: It follows from the cauchy criterion that given $\varepsilon>0$ $\exists$ $H(\varepsilon)$ s.t if $m>n\geq H(\varepsilon)$ then

$$-\varepsilon \leq f_m(x)-f_n(x) \leq \varepsilon, \ \forall x\in[a,b].$$

$$\Rightarrow -\varepsilon(b-a) \leq \int_a^b f_m - \int_a^b f_n \leq \varepsilon(b-a).$$

Since $\varepsilon>0$ is arbitrary, the sequence $\{\int_a^b f_n\}$ is cauchy seqn in $\mathbb{R}$ and therefore converges to some number, say $A\in\mathbb{R}$.

We now show $f\in\mathcal{R}[a,b]$ with integral A. If $\varepsilon>0$ is given, let $K(\varepsilon)$ be such that if $m>K(\varepsilon)$ then $|f_m(x)-f(x)|<\varepsilon$ $\forall x \in[a,b]$. If $\dot{P} := \{([x_{i-1},x_i],t_i)\}_{i=1}^n$ is any tagged partition of $[a,b]$ and if $m>K(\varepsilon)$ then

$$|S(f_m;\dot{P})-S(f;\dot{P})| = \left|\sum_{i=1}^n \{f_m(t_i)-f(t_i)\}(x_i-x_{i-1})\right|$$

$$\leq \sum_{i=1}^n |f_m(t_i)-f(t_i)|(x_i-x_{i-1}).$$

$$\leq \sum_{i=1}^n \varepsilon(x_i-x_{i-1}) = \varepsilon(b-a).$$

We now choose $r\geq K(\varepsilon)$ s.t $|\int_a^b f_r - A|<\varepsilon$ and we let $\delta_{r,\varepsilon}>0$ be such that $|\int_a^b f_r - S(f_r;\dot{P})|<\varepsilon$ whenever $\|\dot{P}\|<\delta_{r,\varepsilon}$. Then we have

$$|S(f;\dot{P})-A| \leq |S(f;\dot{P})-S(f_r;\dot{P})|$$

$$+\left|S(f_r;\dot{P})-\int_a^b f_r\right| + \left|\int_a^b f_r - A\right|$$

$$\leq \varepsilon(b-a)+\varepsilon+\varepsilon = \varepsilon(b-a+2).$$

But since $\varepsilon>0$ is arbitrary, it follows that $f\in\mathcal{R}[a,b]$ and $\int_a^b f = A$.

Th! (Bounded convergence Theorem) Let $\{f_n\}$ be a sequence in $\mathcal{R}[a,b]$ that converges on $[a,b]$ to a function $f\in\mathcal{R}[a,b]$. Suppose also that there exists $B>0$ such that $|f_n(x)|\leq B$ $\forall$ $x\in[a,b]$, $n\in\mathbb{N}$. Then

$$\int_a^b f = \lim_{n\to\infty}\int_a^b f_n$$

(H.T).

(III) Interchange of Limit and continuity

Th: Let $\{f_n\}$ be a sequence of continuous functions on a set $A \subseteq \mathbb{R}$ and suppose that $\{f_n\}$ converges uniformly on A to a function $f: A \to \mathbb{R}$. Then f is continuous on A.

proof: By hypothesis, given $\epsilon > 0$ $\exists$ a natural number $K := K(\frac{1}{3}\epsilon)$ s.t if $n \geq K$, $|f_n(x) - f(x)| < \frac{1}{3}\epsilon$ $\forall x \in A$. (Def^n uniformly convergence)

Let $c \in A$ be arbitrary; we shall show that f is continuous at c. Now

$$|f(x) - f(c)| = |f(x) - f_K(x) + f_K(x) - f_K(c) + f_K(c) - f(c)|$$
$$\leq |f(x) - f_K(x)| + |f_K(x) - f_K(c)| + |f_K(c) - f(c)|$$
$$\leq \tfrac{1}{3}\epsilon + \tfrac{1}{3}\epsilon + |f_K(x) - f_K(c)|.$$

Since f_K is continuous at c, $\exists$ a number $\delta := \delta(\frac{1}{3}\epsilon, c, f_K) > 0$ s.t $|x - c| < \delta$, $x \in A$, then $|f_K(x) - f_K(c)| < \frac{1}{3}\epsilon$.

$\therefore$ $|f(x) - f(c)| \leq \frac{1}{3}\epsilon + \frac{1}{3}\epsilon + \frac{1}{3}\epsilon$, whenever $x \in A$ and $|x - c| < \delta$.

$\Rightarrow$ $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$.

$\therefore$ Since $\epsilon > 0$ is arbitrary, this establishes the continuity of f at the arbitrary point $c \in A$.

Remark: Although the uniform convergence of the sequence of continuous functions is sufficient to guarantee the continuity of the limit function, it is not necessary.

(IV) Interchange of Limit and Derivative:

Th: Let $J \subseteq \mathbb{R}$ be a bounded interval and let $\{f_n\}$ be a sequence of functions on J to $\mathbb{R}$. Suppose that there exists $x_0 \in J$ s.t $\{f_n(x_0)\}$ converges, and that the sequence $\{f_n'\}$ of derivatives exists on J and converges uniformly on J to a function g.

Then the sequence $\{f_n\}$ converges uniformly on J to a function f that has a derivative at every point of J and $f' = g$.