This proves the theorem

* Improper Integrals *

There are two main types of Improper integrals:

(i) The range of integration is unbounded

(ii) The integrand has finite number of infinite discontinuity

Under type (i), there are three kinds of improper integrals.

(a) Let $f(x)$ be bounded and integrable on $a \le x \le B$ for every $B > a$ and $\lim_{B\to\infty} \int_a^B f(x)\,dx$ exists finitely, we then say that the improper integral $\int_a^\infty f(x)\,dx$ is convergent or exists and we write

$$\int_a^\infty f(x)\,dx = \lim_{B\to\infty} \int_a^B f(x)\,dx.$$

(b) Let $f(x)$ be bounded and integrable on $A \le x \le a$ for every $A < a$ and $\lim_{A\to-\infty} \int_A^a f(x)\,dx$ exists finitely, we then say that the improper integral

$\int_{-\infty}^a f(x)\,dx$ exists and we write,

$$\int_{-\infty}^a f(x)\,dx = \lim_{A\to-\infty} \int_A^a f(x)\,dx.$$

(c) Let $f(x)$ be bounded integrable in $A \le x \le a$ for every $A < a$ and in $a \le x \le B$ for every $B > a$, where $A < a < B$. Then the integral $\int_{-\infty}^{\infty} f(x)dx$ is said to convergent if $\int_{-\infty}^{a} f(x)dx$ and $\int_{a}^{\infty} f(x)dx$ are both convergent. We then write

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx$$

$$= \lim_{A \to -\infty} \int_{A}^{a} f(x)dx + \lim_{B \to \infty} \int_{a}^{B} f(x)dx$$

Under type II, there are three kinds of improper integrals.

(i) Let $f(x)$ has infinite discontinuity only at the left end point $x = a$. Then by

$\int_{a}^{b} f(x)dx$ we have $\lim_{\epsilon \to 0^+} \int_{a+\epsilon}^{b} f(x)dx$ when $0 < \epsilon < b-a$

(ii) Let $f(x)$ be bounded and integrable in $a \le x \le b$ and has infinite discontinuity only at the right end point $x = b$.

Then by $\int_{a}^{b} f(x)dx$ we ~~have~~ shall mean $\lim_{\epsilon \to 0^+} \int_{a}^{b-\epsilon} f(x)dx$, when $0 < \epsilon < b-a$.

(iii) Let $f(x)$ has singularity only at $x = c$ and $a < c < b$. Then by $\int_{a}^{b} f(x)dx$ we shall mean $\lim_{\epsilon \to 0^+} \int_{a}^{c-\epsilon} f(x)dx + \lim_{\delta \to 0^+} \int_{c+\delta}^{b} f(x)dx$, $0 < \epsilon < c-a$, $0 < \epsilon < b-c$.

Let $\epsilon = \delta$ then the limit,

$$\lim_{\epsilon \to 0^+} \left[\int_{a}^{c-\epsilon} f(x)dx + \int_{c+\epsilon}^{b} f(x)dx \right]$$

when it exists is called the Cauchy principal value, denoted by $P_c \int_{a}^{b} f(x)dx$

$$\therefore P_c \int_{a}^{b} f(x)dx = \lim_{\epsilon \to 0^+} \left[\int_{a}^{c-\epsilon} f(x)dx + \int_{c+\epsilon}^{b} f(x)dx \right]$$

Problem: prove that the integral $\int_{2}^{\infty} \frac{dx}{x \log x}$ diverges

Solution: Now $\lim_{B \to \infty} \int_{2}^{B} \frac{dx}{x \log x}$

$$= \lim_{B \to \infty} \int_{\log 2}^{\log B} \frac{dz}{z}$$

$= \lim_{B\to\infty} [\log(\log B) - \log(\log 2)]$

Since $\log(\log B)$ increases beyond all bounds and $\log(\log 2)$ is a finite real number, therefore $\int_2^\infty \frac{dx}{x\log x}$ diverges to ∞.

<u>Theorem</u>: If (i) $f(x)$ is bounded integrable in $0 < x < a$ and tends to infinity only when $x \to 0+$ or $f(x)$ is bounded integrable in $0 \le x \le a$ and tends to infinity only when $x \to a$, and (ii) $\int_0^a f(x)\,dx$ converges, then

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$$

<u>Remark</u>: under assumption of the above condition of the theorem

$$\int_0^a f(x)\,dx = \int_0^{a/2} f(x)\,dx + \int_0^{a/2} f(a-x)\,dx$$

<u>problem</u>: Assuming the convergence of the integral of the integrals, prove that

$$\int_0^{\pi/2} \log \sin x\,dx = \int_0^{\pi/2} \log \cos x\,dx = \frac{\pi}{2}\log\frac{1}{2}$$

<u>Solution</u>: Since the integral has been assumed to be convergent, therefore, by

$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, we have

$$\int_0^{\pi/2} \log \sin x\,dx = \int_0^{\pi/2} \log \sin(\pi/2 - x)\,dx = \int_0^{\pi/2} \log \cos x\,dx$$

Let $I = \int_0^{\pi/2} \log \sin x\,dx$. Then

$$2I = \int_0^{\pi/2} \log \sin x\,dx + \int_0^{\pi/2} \log \cos x\,dx$$

$$= \int_0^{\pi/2} \log(\sin x \cos x)\,dx$$

$$= \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right)dx$$

$$= \int_0^{\pi/2} \log \sin 2x\,dx + \log\frac{1}{2}\int_0^{\pi/2} dx$$

$$= \int_0^{\pi/2} \log \sin 2x\,dx + \frac{\pi}{2}\log\frac{1}{2} \quad \text{--- (1)}$$

Now we shall find $\int_0^{\pi/2} \log \sin 2x$. For this we shall evaluate

$\int_{\epsilon}^{\pi/2-\delta} \log \sin 2x \, dx$, when $\epsilon, \delta \to 0^+$.

Let $2x = z$, $dx = \frac{1}{2} dz$

So, $$\int_{\epsilon}^{\pi/2-\delta} \log \sin 2x \, dx = \frac{1}{2}\int_{2\epsilon}^{\pi-2\delta} \log \sin z \, dz$$

$$= \frac{1}{2} \int_{2\epsilon}^{\pi-2\delta} \log(\sin x)\, dx$$

Letting $\epsilon, \delta \to 0^+$, we get

$$\int_0^{\pi/2} \log(\sin 2x)\, dx = \frac{1}{2}\int_0^{\pi} \log(\sin x)\, dx$$

$$= \frac{1}{2}\cdot 2 \int_0^{\pi/2} \log(\sin x)\, dx \quad \text{by a known result}$$

$$= I$$

Thus from (i) $\quad 2I = I + \frac{\pi}{2}\log\frac{1}{2}$

or, $I = \frac{\pi}{2} \log 1/2$

i.e. $$\int_0^{\pi/2} \log \sin x \, dx = \int_0^{\pi/2} \log \cos x \, dx = \frac{\pi}{2}\log\frac{1}{2}$$

Theorem:- necessary and sufficient condition for Improper integral.

Definition:- Let $f(x)$ be bounded and integrable in $a \le x \le B$ for every $B > a$. Then $\int_a^{\infty} f(x)\,dx$ is said to converge to I if for any given $\epsilon > 0$, $\exists$ a positive real number X such that

$$\left| I - \int_a^{B} f(x)\,dx \right| < \epsilon \quad \text{for every } B > X.$$

Theorem:- (Cauchy's Criterion) A necessary and sufficient condition for the improper integral $\int_a^{\infty} f(x)\,dx$ (point) is that for every preassigned $\epsilon > 0$, $\exists$ a positive real number X such that $\left| \int_{x'}^{x''} f(x)\,dx \right| < \epsilon$ for $x'' > x' > X$

Definition:- The improper integral $\int_a^{\infty} f(x)\,dx$ where $f(x)$ is bounded integrable in $a \le x \le B$ for every $B > a$ is said to converge absolutely if $\int_a^{\infty} |f(x)|\,dx$ converges. But if the first integral converges and the second diverges then we say that $\int_a^{\infty} f(x)\,dx$ converges conditionally.

Theorem: Every absolutely convergent integral is convergent.

proof: Let $\int_a^{\infty} |f(x)|\,dx$ converge finitely. Let $\epsilon>0$ be preassigned. Then by Cauchy's criterion $\exists$ a positive real number x such that

$$\int_{x'}^{x''} |f(x)|\,dx < \epsilon \quad \text{for } x''>x'>x.$$

$$\text{So, } \left|\int_{x'}^{x''} f(x)\,dx\right| \leq \int_{x'}^{x''} |f(x)|\,dx < \epsilon \quad \text{for } x''>x'>x.$$

Hence $\int_a^{\infty} f(x)\,dx$ converges.

problem: prove that $\int_0^{\infty} \frac{\sin x}{x}\,dx$ converges but not absolutely.

Solution: Since, $\lim_{x\to 0} \frac{\sin x}{x} = 1$, $x=0$ is not a singularity of $\frac{\sin x}{x}$. Now, let $f(x) = \frac{1}{x}$ and $\phi(x) = \sin x$, $x \in [x', x'']$, $x''>x'>0$. Then $f(x)$ and $\phi(x)$ are bounded integrable on $[x', x'']$ and $f(x)$ is monotonic on $[x', x'']$. Thus by 2nd mean value theorem (due to Weierstrass), we have

$$\int_{x'}^{x''} \frac{\sin x}{x}\,dx = \frac{1}{x'}\int_{x'}^{\xi} \sin x\,dx + \frac{1}{x''}\int_{\xi}^{x''} \sin x\,dx, \quad x' \leq \xi \leq x''$$

Now $\left|\int_{x'}^{\xi} \sin x\,dx\right| = |\cos x' - \cos \xi| \leq |\cos x'| + |\cos \xi| \leq 2$ and

similarly $\left|\int_{\xi}^{x''} \sin x\,dx\right| \leq 2$.

$$\therefore \left|\int_{x'}^{x''} \frac{\sin x}{x}\,dx\right| \leq \frac{1}{x'}\left|\int_{x'}^{\xi} \sin x\,dx\right| + \frac{1}{x''}\left|\int_{\xi}^{x''} \sin x\,dx\right|$$

$$\leq 2\left(\frac{1}{x'} + \frac{1}{x''}\right) < \frac{4}{x'}.$$

Let $\epsilon>0$ be given then

$$\left|\int_{x'}^{x''} \frac{\sin x}{x}\,dx\right| < \frac{4}{x'} < \epsilon \quad \text{~~therefore~~ for } x''>x'>X$$

where $X = \frac{4}{\epsilon}$

Hence $\int_0^{\infty} \frac{\sin x}{x}\,dx$ converges.

Now we shall show that $\int_0^{\infty} \left|\frac{\sin x}{x}\right| dx = \int_0^{\infty} \frac{|\sin x|}{x}\,dx$ diverges to ∞.

For this we shall consider the integral $\int_0^{n\pi} \frac{|\sin x|}{x}\,dx$, where n is a positive integer. Then

$$\int_0^{n\pi} \frac{|\sin x|}{x}\, dx = \sum_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{x}\, dx$$

Let, $x = (r-1)\pi + t$, then $dx = dt$ and t varies from 0 to π as x varies from $(r-1)\pi$ to $r\pi$. Thus,

$$\int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{x}\, dx = \int_0^{\pi} \frac{|\sin\{(r-1)\pi + t\}|}{(r-1)\pi + t}\, dt$$

$$= \int_0^{\pi} \frac{|(-1)^{r-1}\sin t|}{(r-1)\pi + t}\, dt$$

$$= \int_0^{\pi} \frac{|\sin t|}{(r-1)\pi + t}\, dt$$

Since $|\sin\{(r-1)\pi + t\}| = |(-1)^{r-1}\sin t| = |\sin t|$ & $0 \le t \le \pi$

Thus $\displaystyle\int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{x}\, dx = \int_0^{\pi} \frac{\sin t}{(r-1)\pi + t}\, dt$

But $(r-1)\pi + t$ is maximum at $t = \pi$ & its maximum value is $(r-1)\pi + \pi = r\pi$.

$$\therefore \int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{x}\, dx \geqslant \frac{1}{r\pi} \int_0^{\pi} \sin t\, dt$$

$$= \frac{1}{r\pi}\left[-\cos t\right]_0^{\pi}$$

$$= \frac{2}{r\pi}$$

So, $\displaystyle\int_0^{n\pi} \frac{|\sin x|}{x}\, dx \geqslant \sum_{r=1}^{n} \frac{2}{r\pi} = \frac{2}{\pi}\sum_{r=1}^{n} \frac{1}{r}$

But the series on the right side diverges to ∞ as $n \to \infty$. So,

$\displaystyle\int_0^{n\pi} \frac{|\sin x|}{x}\, dx \to \infty$ as $n \to \infty$.

Let X be a real number. Then without loss of generality we may assume that $n\pi \le X < (n+1)\pi$, where n is a positive integer. Then,

$$\int_0^{X} \frac{|\sin x|}{x}\, dx \geqslant \int_0^{n\pi} \frac{|\sin x|}{x}\, dx \text{ and as } X \to \infty, n \to \infty.$$

So, $\displaystyle\int_0^{X} \frac{|\sin x|}{x}\, dx \to \infty$ as $X \to \infty$, i.e;

$$\int_0^{\infty} \frac{|\sin x|}{x}\, dx \text{ diverges to } \infty.$$

prove that the integral $\displaystyle\int_{-\infty}^{\infty} x e^{-x^2}\, dx$ converges

Solution:- Now, $\displaystyle\int_{-\infty}^{\infty} x e^{-x^2}\, dx = \int_{-\infty}^{0} x e^{-x^2}\, dx + \int_0^{\infty} x e^{-x^2}\, dx$

$$= \lim_{A \to -\infty} \int_A^0 x e^{-x^2}\, dx + \lim_{B \to \infty} \int_0^B x e^{-x^2}\, dx$$

$$= \lim_{A\to-\infty} -\frac{1}{2}\int_A^0 -2x e^{-x^2} dx + \lim_{B\to\infty} \frac{-1}{2}\int_0^B -2xe^{-x^2} dx$$

$$= \lim_{A\to-\infty} -\frac{1}{2}[e^{-x^2}]_A^0 + \lim_{B\to\infty} -\frac{1}{2}[e^{-B^2}-1]$$

$$= -\frac{1}{2} + \frac{1}{2}.$$

$$= 0$$

So, $\int_{-\infty}^{\infty} xe^{-x^2} dx$ converges to 0.

Theorem:- If $f(x)$ and $g(x)$ be integrable functions when $x \geq a$ such that $0 \leq f(x) \leq g(x)$ for $x \geq a$, then

(i) $\int_a^\infty f(x) dx$ converges if $\int_a^\infty g(x) dx$ converges

and (ii) $\int_a^\infty g(x) dx$ diverges if $\int_a^\infty f(x) dx$ diverges

Comparison integral I

The integral $\int_0^\infty e^{-px} dx$, where p is a constant converges for $p>0$ & diverges for $p \leq 0$

proof:- We have, for $B>0$,

$$\int_0^B e^{-px} dx = -\frac{1}{p}[e^{-pB}-1] = \frac{1}{p}(1-e^{-pB}) \text{ when } p \neq 0$$

and $\int_0^B dx = B$, when $p=0$

Letting $B\to\infty$; we have

$$\int_0^\infty e^{-px} dx = \frac{1}{p}, \text{ when } p>0$$

and $\int_0^\infty e^{-px} dx$ diverges when $p \leq 0$

So, $\int_0^\infty e^{-px} dx$, p is a constant, converges for $p>0$ and diverges for $p\leq 0$

Comparison Integral II

The improper integral $\int_a^\infty \frac{dx}{x^\mu}$ $(a>0)$ exists for $\mu>1$ and does not exist for $\mu \leq 1$

proof:- We have for every $B>a$.

$$\int_a^B \frac{dx}{x^\mu} = \frac{1}{1-\mu}[B^{1-\mu} - a^{1-\mu}], \mu \neq 1$$

and $\int_a^B \frac{dx}{x} = \log B - \log a$, when $\mu=1$

letting $B\to\infty$, we have

$$\int^\infty \frac{dx}{} = \frac{a^{1-\mu}}{\mu-1}, \text{ when } \mu>1$$

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