(1)

Vector Space

A non-empty set V is said to form a real vector space (or a vector space over the field $\mathbb{R}$) if

(i) there is a binary composition $(+)$ on V, called addition satisfying the conditions.

V_1. $\alpha + \beta \in V$ for all $\alpha, \beta \in V$

V_2. $\alpha + \beta = \beta + \alpha$ $\forall \alpha, \beta \in V$

V_3. $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$ $\forall \alpha, \beta, \gamma \in V$

V_4 There exist an element θ in V such that $\alpha + \theta = \alpha$ $\forall \alpha \in V$.

V_5. For each α in V there exists an element $-\alpha$ in V such that $\alpha + (-\alpha) = \theta$.

and (ii) there is an external composition of $\mathbb{R}$ with V, called multiplication satisfying the following conditio

2020/4/7 09:04

(2)

V_6. $c\alpha \in V$ $\forall c \in R$, all $\alpha \in V$

V_7. $c(d\alpha) = (cd)\alpha$ $\forall c, d \in R$ and $\forall \alpha \in V$

V_8. $c(\alpha+\beta) = c\alpha + c\beta$ $\forall c \in R$ and $\forall \alpha, \beta \in V$.

V_9. $(c+d)\alpha = c\alpha + d\alpha$ $\forall c, d \in R$ and $\forall \alpha \in V$.

V_{10}. $1\alpha = \alpha$, 1 being the identity element in R.

Theorem: In a vector space V over a field F.

(i) $0\alpha = \theta$ $\forall \alpha \in V$
(ii) $c\theta = \theta$ $\forall c \in F$
(iii) $-1\alpha = -\alpha$ $\forall \alpha \in V$, 1 being the identity in F
(iv) $c\alpha = \theta \Rightarrow$ either $c=0$ or $\alpha=0$

Proof: (i) 0 is the zero element in F

$\therefore$ ~~$0 = 0+0$ in F~~ $0+0 = 0$ in F
$\Rightarrow (0+0)\alpha = 0.\alpha$ in V
$\Rightarrow 0\alpha + 0\alpha = 0\alpha$, by V_9
$-0\alpha \in V$ $\because 0\alpha \in V$

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JANUARY '17

2020/4/7 09:04

Therefore $-0\alpha + (0\alpha + 0\alpha) = -0\alpha + 0\alpha$

$\Rightarrow (-0\alpha + 0\alpha) + 0\alpha = \theta$ by V_3 and V_5

$\Rightarrow \theta + 0\alpha = \theta$ by V_5

$\Rightarrow 0\alpha = \theta$ by V_4

(ii) θ is the zero element in V.

$\therefore \theta + \theta = \theta$ in V.

$\Rightarrow c(\theta + \theta) = c\theta$.

$\Rightarrow c\theta + c\theta = c\theta$ by V_8

$-c\theta \in V \quad \because c\theta \in V$

Therefore $-c\theta + (c\theta + c\theta) = -c\theta + c\theta$

$\Rightarrow (-c\theta + c\theta) + c\theta = \theta$ by V_3 & V_5

$\Rightarrow \theta + c\theta = \theta$ by V_5

$\Rightarrow c\theta = \theta$ by V_4

(iii) We have

$$\theta = 0\alpha$$
$$= [1 + (-1)]\alpha$$
$$= 1\alpha + (-1)\alpha$$
$$= \alpha + (-1)\alpha \quad \text{by } V_{10}$$

Therefore $-\alpha + \theta = -\alpha + \{\alpha + (-1)\alpha\}$

$$= (-\alpha + \alpha) + (-1)\alpha \quad [\text{by } V_3$$
$$= \theta + (-1)\alpha \quad \text{by } V_5$$
$$\therefore \quad -\alpha = (-1)\alpha \quad \text{by } V_4$$

(iv) $c\alpha = \theta$ and let $c \neq 0$ then c^{-1} exists in F

Now $c\alpha = \theta \Rightarrow c^{-1}(c\alpha) = c^{-1}\theta$

$$\Rightarrow (c^{-1}c)\alpha = c^{-1}\theta \quad \text{by } V_7$$
$$\Rightarrow 1.\alpha = \theta \quad \text{by (ii)}$$
$$\Rightarrow \alpha = \theta \quad \text{by } V_{10}.$$

Therefore $c\alpha = \theta$ and $c \neq 0 \Rightarrow \alpha = \theta$

Contrapositively $c\alpha = \theta$ and $\alpha \neq 0 \Rightarrow c = 0$

Hence $c\alpha = \theta \Rightarrow$ either $c = 0$ or $\alpha = \theta$.

Subspace :- Let V be a vector space over a field F.

Let W be a non-empty subset of V.

If W forms a vector space over F with respect to $\oplus$ and $\odot$ Then W is said to be a subspace of V.

Th:- A non-empty subset W of a vector space V over a field F is a subspace of V if and only if

(i) $\alpha \in W, \beta \in W \Rightarrow \alpha+\beta \in W$

(ii) $\alpha \in W, c \in F \Rightarrow c\alpha \in W$.

otherwise $a\alpha + b\beta \in W \quad \forall \alpha, \beta \in W$ & $a, b \in F$.

Ex. Let S be the subset of $\mathbb{R}^3$ defined by $S = \{(x, y, z) \in \mathbb{R}^3 : y = z = 0\}$

Examine S is a subspace of $\mathbb{R}^3$

Sol^n: $(0, 0, 0) \in S$

$\therefore S$ is a non-empty subset of $\mathbb{R}^3$

Let $\alpha = (x_1, 0, 0) \in S$

$\beta = (x_2, 0, 0) \in S$

$\therefore$ Let $c, d \in \mathbb{R}$

$\therefore c\alpha + d\beta = c(x_1, 0, 0) + d(x_2, 0, 0)$

$= (cx_1 + dx_2, 0, 0) \in S$

$\because cx_1 + dx_2 \in \mathbb{R}$

This proves that S is a subspace of $\mathbb{R}^3$.

$\Rightarrow$ S is non-empty subset of $\mathbb{R}^3$ $\because (0,0,1) \in S$

but $\theta \notin S$

$\therefore$ S is not a subspace of $\mathbb{R}^3$

Theorem:— The intersection of two subspace of a vector space V over a field F is a subspace of V.

Proof: Let W_1 and W_2 be two subspace of V.

$W_1 \cap W_2$ is not empty because $\theta \in W_1 \cap W_2$

Case 1, Let $W_1 \cap W_2 = \{\theta\}$ then $W_1 \cap W_2$ is subspace of V.

Case 2: Let $W_1 \cap W_2 \neq \{\theta\}$ and let $\alpha_1 \in W_1 \cap W_2$ and $\alpha_2 \in W_1 \cap W_2$

Then $\alpha_1, \alpha_2 \in W_1$ and $\alpha_1, \alpha_2 \in W_2$

since W_1 is a subspace of V, (i) $\alpha_1 + \alpha_2 \in W_1$

(ii) $c\alpha \in W_1$, $c \in \mathbb{R}$

SUNDAY 22

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FEBRUARY '17

Again since W_2 is a subspace of V

(i) $\alpha_1 + \alpha_2 \in W_2$

(ii) $c\alpha_1 \in W_2$, $c \in \mathbb{R}$

Therefore $\alpha_1 + \alpha_2 \in W_1 \cap W_2$ and $c\alpha_1 \in W_1 \cap W_2$

This proves that $W_1 \cap W_2$ is a subspace of V.

(*) But Union of two subspace of V is not, in general, a subspace of V.

For example:- Let us consider two subspace S and T of the vector space $\mathbb{R}^3$

Where $S = \{(x, y, z) \in \mathbb{R}^3 : y = 0, z = 0\}$

$T = \{(x, y, z) \in \mathbb{R}^3 : x = 0, z = 0\}$

Let $\alpha = (1, 0, 0) \in S$ and $\beta = (0, 1, 0) \in T$

$\therefore \alpha + \beta = (1, 1, 0)$

Here ~~$\alpha + \beta \notin S$ and $\alpha + \beta \notin T$~~

$\therefore$ ~~$\alpha + \beta \notin S \cup T$~~

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JANUARY '17