Travelling Salesman Problem – A Review

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Abstract:

In the present talk we will review the traveling salesman problem. The method is not an exact one and optimal solution is obtained only after finding all the tours which becomes unrealistic for large number of cities.

The problem can also be reformulated as one of finding the minimal Hamiltonian circuit from a complete weighted graph of n vertices. Here we will consider in details this alternative approach of solving a traveling salesman problem with suitable example. A simple method known as the nearest neighbour method will be discussed that gives reasonably good result. It may be noted that we know of no efficient procedure for solving the problem exactly. However lower bound may be provided for the length of the route.

Introduction

The traveling salesman problem was first formulated as a mathematical problem in 1930 and is one of the most intensively studied problems in optimization. It is used as a benchmark for many optimization methods. In a travelling salesman problem (TSP) a salesman is required to visit each of the town in which every pair of town is connected by roads. It is also require to return to the starting town after visiting each of the towns exactly once in such a way that the total distance travelled by him is minimum. So the problem is to plan a round trip to visit each of the town exactly once and, if such a trip is possible, can he plan one which minimizes the total distance traveled? It may be noted that a symmetric TSP for 15 towns has (14!)/2 = 43 589 145 600 possible tours visiting each city exactly once.

Several heuristic methods of solution are available to give a route very close to the shortest one, but do not guarantee the shortest. If the total number of town is n and each of the town is visited once only then starting from a given town the salesman will have before him a total $(n-1)!$ different sequence of possible routes if the problem be a non symmetric and if symmetric then it will be $(n-1)!/2$. So the problem of traveling salesman can be solved by calculating the distance traveled in each of the route and then picking the shortest one. However for a large value of n, the labor involved is too great even for a computer.

The TSP has several applications even in its purest formulation, such as planning, logistics, and the manufacture of microchips. Slightly modified, it appears as a sub-problem in many areas, such as DNA sequencing. In these applications, the concept city represents, for example, customers, soldering points, or DNA fragments, and the concept distance represents travelling times or cost, or a similarity measure between DNA fragments. In many applications, additional constraints such as limited resources or time windows make the problem considerably harder.

Formulation of the problem

With some differences a Travelling salesman problem can be formulated as an assignment problem.

Let c_{ij} denote the distance as the salesman goes from city i to city j and let,

$x_{ij}=1$, if the salesman goes directly from city i to the city j

$=0$, otherwise.

As x_{ii} can never be unity so by convention $c_{ii}=\infty$

Then the problem is

$$\text{Min. } z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}$$

$$\text{such that } \sum_{j=1}^{n} x_{ij} = 1, \quad i = 1,2,\ldots,n$$

$$\text{and, } \sum_{i=1}^{n} x_{ij} = 1, \quad j = 1,2,\ldots,n.$$

Let us consider one example:

	A	B	C	D
A	∞	17	15	10
B	50	∞	25	44
C	45	20	∞	40
D	20	19	18	∞

We can solve this problem by exhaustively enumerating the (4-1)!=6 possible loops of the network. One can get $A-D-C-B-A$ as the optimum loop. But exhaustive enumeration is practical only for small values of n. One can approach to solve it as an Assignment problem. Then also he will reach to find the assignment as A-D, B-C, C-B, D-A. But to satisfy the conditions of Travelling salesman problem he must have to use trial and error method. Only a lower bound of the value can be found if one solves the problem as an Assignment problem. One can never be confirm about the optimal path until he judge all the possibilities.

Thus we see that the Travelling Salesman problem turns out to be a difficult one in that we know of no efficient procedure for solving the problem. One might wish to look for simple procedures that will give good results to the problem.

Alternatively one can think the problem as graph theoretically.

We can represent the salesman's territory by a weighted graph G where the vertices corresponds to the towns and each pair of vertices are joined by a weighted edge corresponding to the distance(time or cost) to travel between the towns.

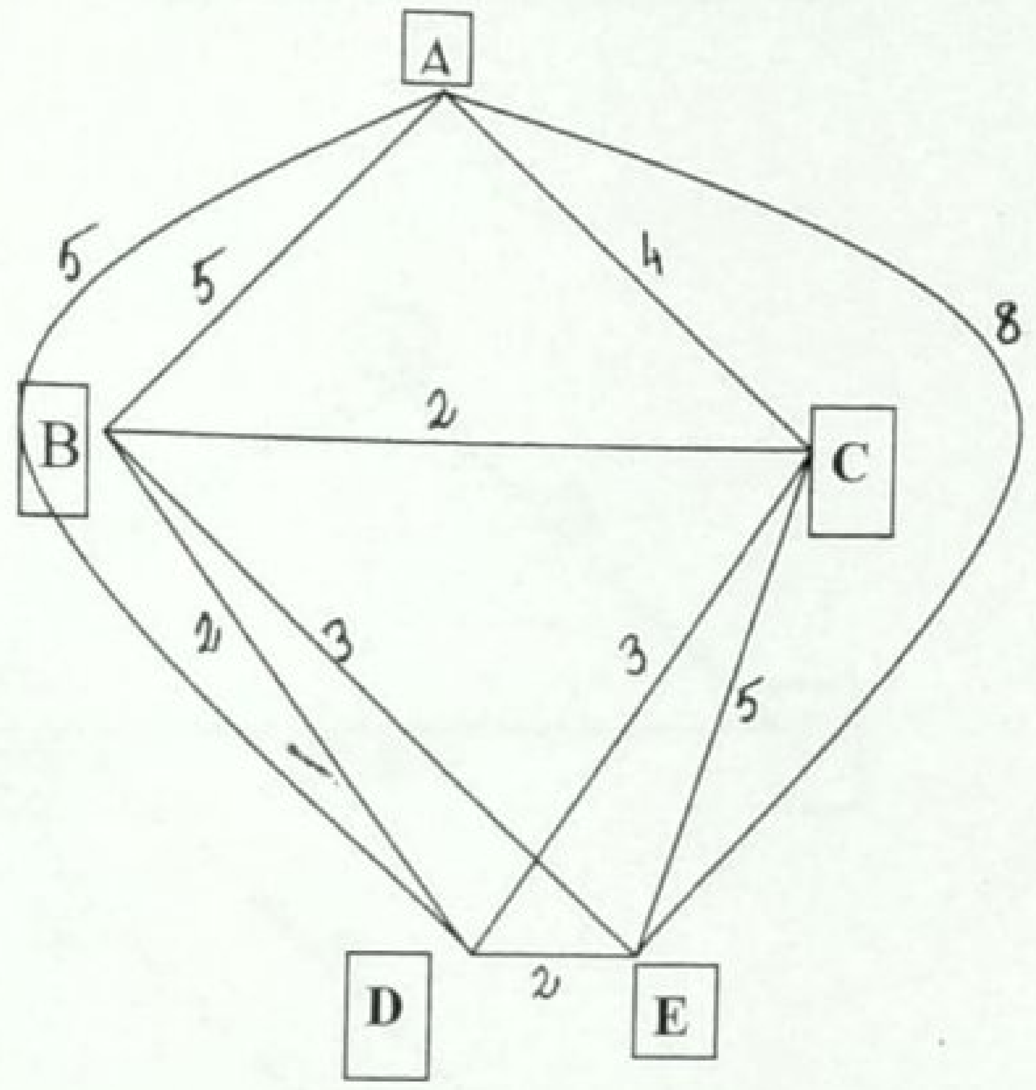
Thus we have a complete weighted graph and the traveling salesman problem asks for a Hamiltonian circuit of minimum weight. There are two difficulties that arise with the problem:

i) It is sometimes difficult to determine if a graph is Hamiltonian and,
ii) given a weighted graph G which is Hamiltonian there is no easy or efficient algorithm for finding an optimal circuit in G, in general.

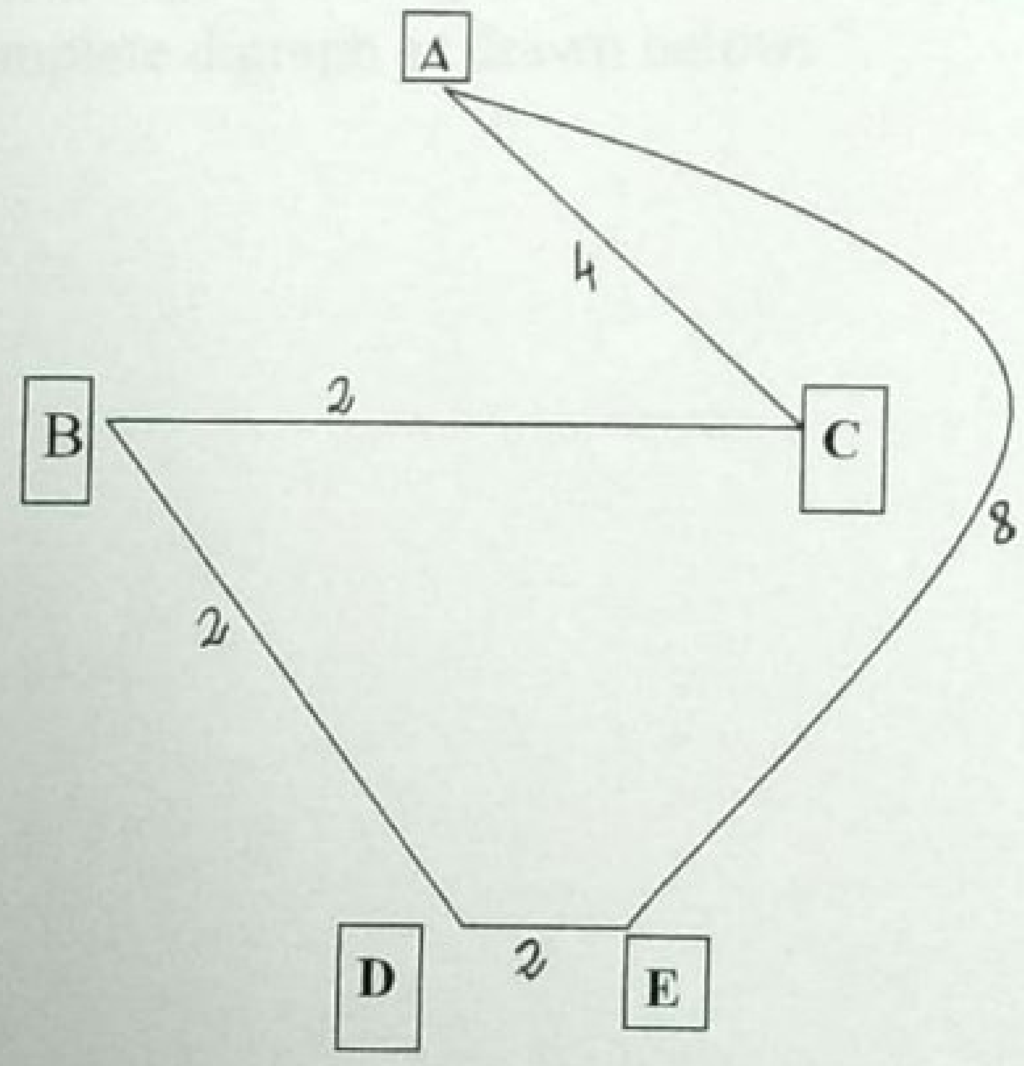
There is a procedure known as the nearest neighbor method which gives reasonably good result for the Travelling Salesman problem.

i) start with an arbitrary chosen vertex and find the vertex that is closest to the starting vertex to form an initial path of one edge.
ii) Let x denote the latest vertex that was added to the path. Among all the vertices that are not in the path, pick the one that is closest to x and to the path the edge connecting x and this vertex.
iii) Form a circuit by adding the edge connecting the starting vertex and the last vertex added.

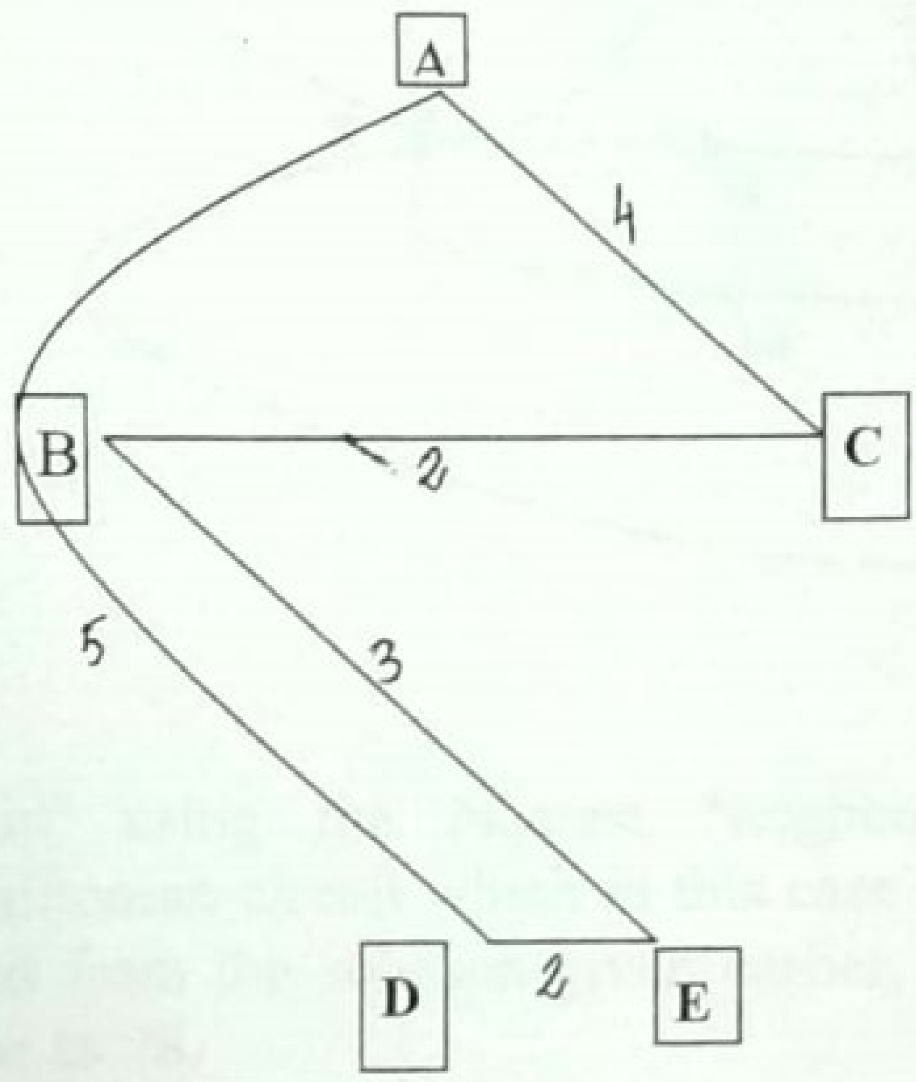
Let us illustrate this algorithm first by taking a symmetric TSP as follows:



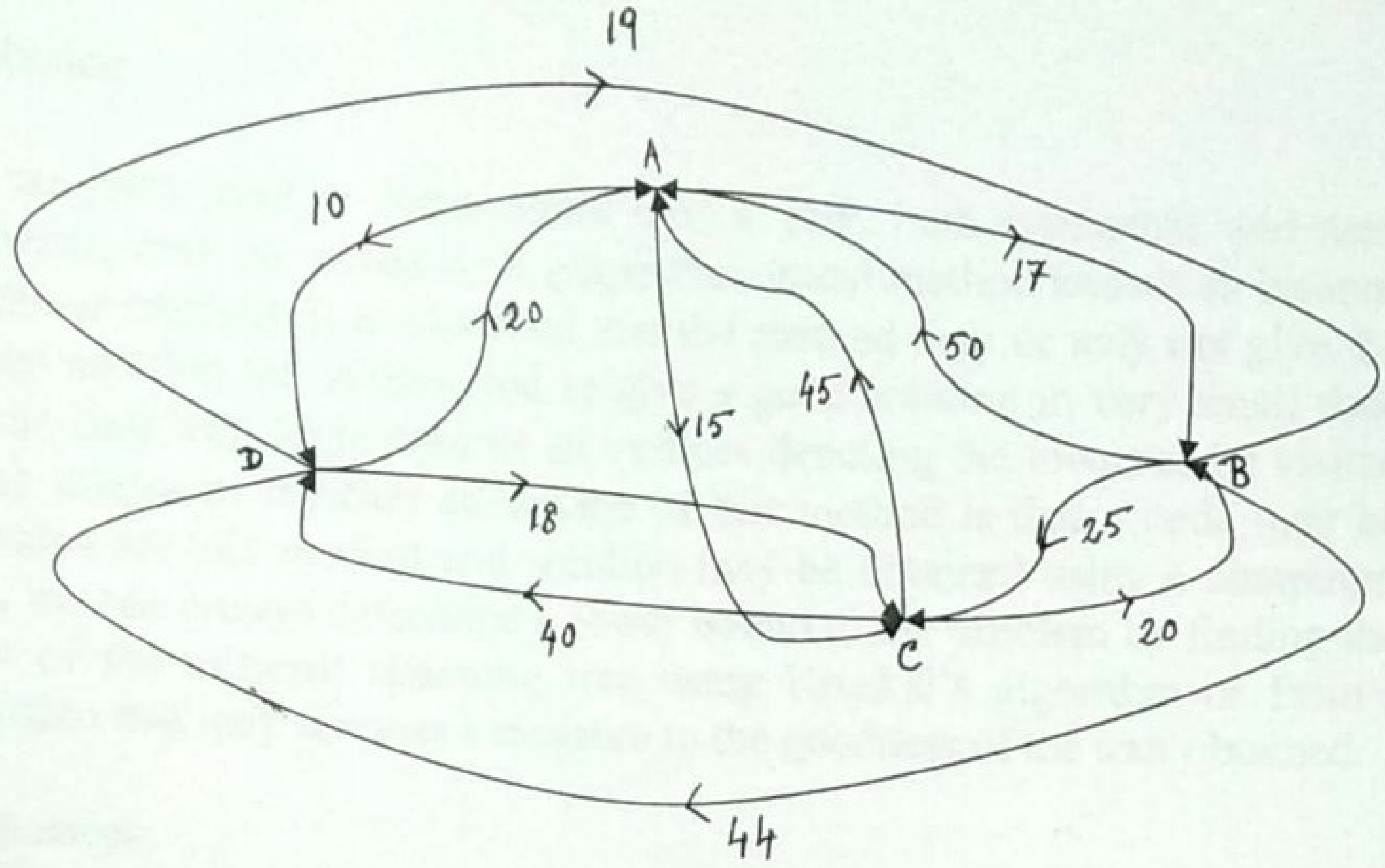
Starting from the vertex A and applying the Nearest Neighbor Method one can get the Hamiltonian circuit A-C-B-D-E-A as a good tour with the value of the TSP as 18 unit. The following graph represents this Hamiltonian circuit:



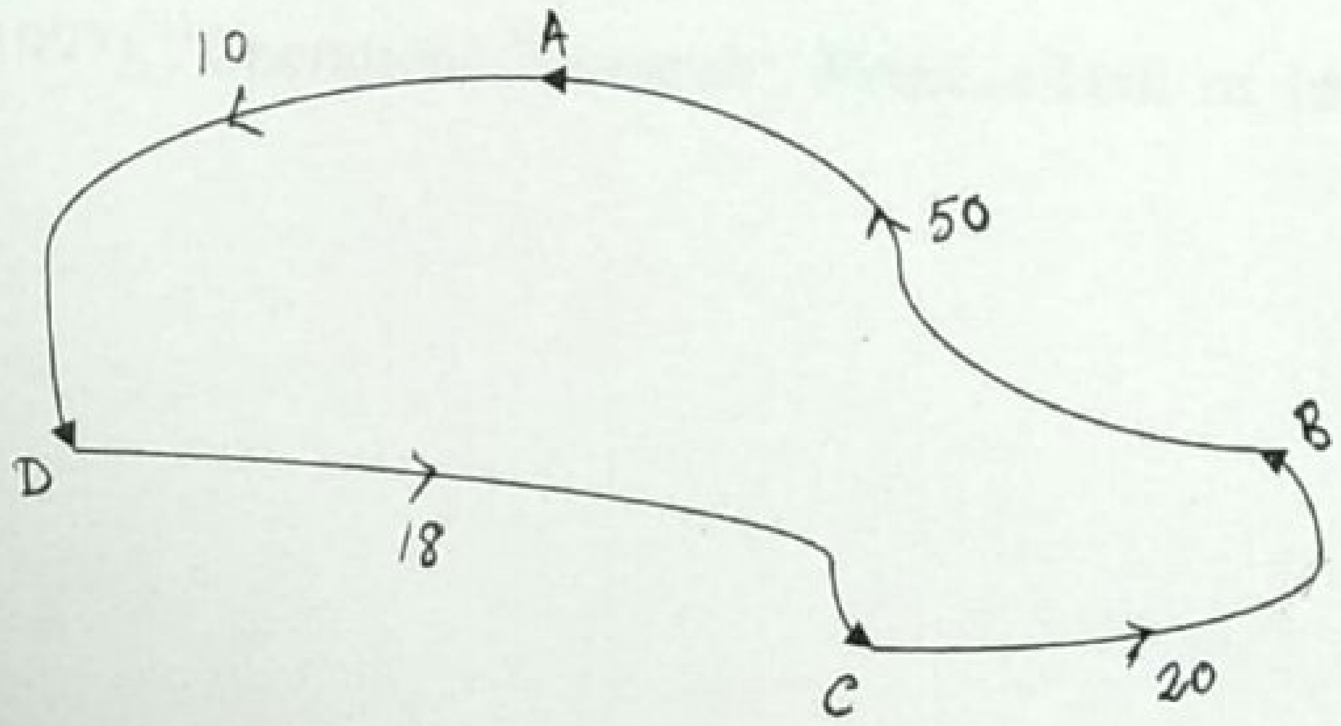
But, it may be noted that this is not the minimal Hamiltonian circuit which is the following with its value as 16 unit:



Now let us consider the graph of the TSP given earlier in tabular form, which was a non-symmetric TSP. For this problem we get an equivalent complete digraph as drawn below:



Again, using the Nearest Neighbor Method we get the following Hamiltonian circuit which in this case is indeed the minimal one, as may be noted from the solution given earlier, with the tour as A-D-C-B-A having value as 98.



Conclusion

Here we have tried to demonstrate how a TSP, both symmetric and non-symmetric, may be solved by a graph-theoretical method known as Nearest Neighbour Method. It is observed that the method may or may not give the optimal solution but is expected to give a good solution in very small time and can deal with large number of vertices denoting the towns to be visited by the salesman. Another advantage of this method is that a code may be generated for this method and solution may be obtained using a computer. Also, we can always determine a lower bound to the problem by finding the value of the minimal spanning tree using Kruskal's algorithm or Prim's algorithm that may serve as a measure to the goodness of the tour obtained.

References

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