CHAPTER-4

Continuity

Definition: Let $f:(X,d)\to(Y,\rho)$ be a function and $c\in X$. Then f is said to be continuous at $x=c$ if for a given $\epsilon>0$, $\exists$ a $\delta>0$ such that

$$\rho(f(x),f(c))<\epsilon \text{ whenever } d(x,c)<\delta.$$

i.e., $f(x)\in S_\epsilon(f(c))$, whenever ~~d(x,c) < δ~~ $x\in S_\delta(c)$

i.e., $f(S_\delta(c))\subseteq S_\epsilon(f(c))$

Theorem: Let $f:(X,d)\to(Y,\rho)$ ~~be continuous at x=c ∈ X~~ a function. Then f is continuous at $x=c$ in X iff for every sequence $\{x_n\}$ with $\lim_{n\to\infty} x_n=c$, the sequence $\{f(x_n)\}$ is convergent and $\lim_{n\to\infty} f(x_n)=f(c)=f(\lim_{n\to\infty} x_n)$

proof: Let $f:(X,d)\to(Y,\rho)$ be continuous at $x=c\in X$ and $\{x_n\}$ be a sequence in (X,d) with $\lim_{n\to\infty} x_n=c$. Let $\epsilon>0$ be given, then due to continuity of $f(x)$ at $x=c$, $\exists$ a $\delta>0$ such that

$$\rho(f(x),f(c))<\epsilon \text{ whenever } d(x,c)<\delta. \quad ---(1)$$

Since, $\lim_{n\to\infty} x_n=c$, corresponding to above $\delta>0$ $\exists$ a positive integer n_0 such that

$$d(x_n,c)<\delta,\ \forall n\geq n_0 \quad --(2)$$

Using (2) in (1), we have

$$\rho(f(x_n),f(c))<\epsilon,\ \forall n\geq n_0.$$

This shows that $\lim_{n\to\infty} f(x_n)=f(c)=f(\lim_{n\to\infty} x_n)$

Conversely suppose that the condition of the above theorem holds, but f fails to be continuous at $x=c$. Then $\exists$ an $\epsilon>0$, say ϵ_0 such that for every positive δ, there is a point x depending on δ such that

$$\rho(f(x_n), f(c)) \geq \epsilon_0 \quad \text{although} \quad d(x,c)<\delta$$

Taking $\delta = 1, 1/2, 1/3, \ldots, 1/n, \ldots$ in succession, we have points $x_1, x_2, \ldots, x_n, \ldots$ such that

$$\rho(f(x_n), f(c)) \geq \epsilon_0 \quad \text{although} \quad d(x_n,c) \leq 1/n \text{ for all } n=1,2,\ldots$$

This shows that $f(x_n) \not\to f(c)$ as $n\to\infty$ although $x_n \to c$ as $n\to\infty$ which contradicts our assumption. Hence $f(x)$ must be continuous at $x=c$.

Definition: If $f:(X,d)\to(Y,\rho)$ is continuous at all points of X. Then f is said to be continuous on X.

Theorem: A function $(X,d)\to(Y,\rho)$ is continuous on X iff $f^{-1}(G)$ is an open set in (X,d) whenever G is an open set in (Y,ρ).

proof: Let $f:(X,d)\to(Y,\rho)$ be continuous on X and let G be an open set in (Y,ρ). If $G=\phi$ or $G=Y$, then $f^{-1}(G)=\phi$ or X respectively and so in this case $f^{-1}(G)$ is open. So suppose $\phi\neq G\neq Y$. Let $E=f^{-1}(G)$ and $u\in E$ then $f(u)\in G$. Since G is open $\exists$ an open ball $S_\epsilon(f(u))$ such that $S_\epsilon(f(u))\subset G$. Since f is continuous at $u\in X$. Corresponding to $\epsilon>0$, $\exists$ a $\delta>0$ such that

$$f(S_\delta(u)) \subseteq S_\epsilon(f(u)) \subseteq G$$

i.e., $S_\delta(u) \subseteq f^{-1}(G)$

Since u is arbitrary this shows that every point of $f^{-1}(G)$ is an interior point of $f^{-1}(G)$.

Thus $f^{-1}(G)$ is an open set in (X,d).

Conversely suppose that condition of the theorem holds. Let $u\in X$ and $\epsilon>0$ be given. Consider the open ball $S_\epsilon(f(u))$, which is an open set in (Y,ρ). So, by hypothesis $f^{-1}(S_\epsilon(f(u)))$ is an open set in (X,d) and $u\in f^{-1}(S_\epsilon(f(u)))$. Since u is an interior point of $f^{-1}(S_\epsilon(f(u)))$ there exist a $\delta>0$ such that $S_\delta(u)\subset f^{-1}(S_\epsilon(f(u)))$. Therefore $f(S_\delta(u))\subset S_\epsilon(f(u))$

i.e, $\rho(f(x), f(u))<\epsilon$ whenever $d(x,u)<\delta$.

Therefore f is continuous at $u\in X$. Since u is arbitrary point of X it follows that f is continuous on X.

Corollary: A function $f:(X,d)\to(Y,\rho)$ is continuous on X iff $f^{-1}(G)$ is a closed set in (X,d) whenever G is closed in (Y,ρ).

Definition: A function $f:(X,d)\to(Y,\rho)$ is said to be uniformly continuous on X if for any given $\epsilon>0$ $\exists$ a $\delta>0$ such that $\rho(f(x),f(x'))<\epsilon$, whenever $d(x,x')<\delta$.

Theorem: If $f:(X,d)\to(Y,\rho)$ is uniformly continuous then f is continuous on X.

Proof: Let $\epsilon>0$ be preassigned. Since f is uniformly continuous on X, $\exists$ a $\delta>0$ such that

$$\rho(f(x),f(x'))<\epsilon,\ \text{whenever } d(x,x')<\delta.$$

Let u be any point of X. Taking $x'=u$ we have

$\rho(f(x),f(u))<\epsilon$ whenever $d(x,u)<\delta$

Therefore f is continuous at $u\in X$. Since u is arbitrary, it follows that f is continuous on X.

~~Then~~ Whenever the converse of this theorem is not true. Let $X=Y=$ the space of real numbers with respect to the usual metric 'd'. Let $f:R\to R$ be defined by

$f(x)=x^2,\ x\in R$. Then $f(x)$ being a polynomial is continuous on R. Let us take $\epsilon=1$. If f is uniformly continuous on R then $\exists$ a $\delta>0$ such that $|f(x)-f(x')|<1$ whenever $d(x,x')<\delta$. But then for $x\in R$, we have

$$|x+\tfrac{1}{2}\delta-x|=\tfrac{1}{2}\delta<\delta \text{ and}$$

$$|f(x')-f(x)| = |(x+\tfrac{1}{2}\delta)^2-x^2|$$

$$= |x^2+\delta x+\tfrac{1}{4}\delta^2-x^2|$$

$$= \frac{\delta}{4}|4x+\delta|,$$ which can be increased sufficiently large by taking x suitably large. Hence, f fails to be uniformly continuous on X.