Date: 28-08-20

Integral Surface passing through a Given Curve:

In the previous section, we have seen how a general solⁿ for a given linear PDE can be obtained. Now, we shall make us of this general solⁿ to find an integral surface containing a given curve as explained below:

Suppose $ap+bq=c$ be a eqⁿ and $f(\varphi, \psi)=0$ is solⁿ of this eqⁿ with

$$\varphi(x, y, u)=c_1 \text{ and } \psi(x, y, u)=c_2$$

where
$$\left.\begin{aligned}\varphi(x,y,u)=c_1\\ \psi(x,y,u)=c_2\end{aligned}\right\}$$

are two integral curves obtain from the auxilary eqⁿ of a give PDE.

Suppose, we wish to determine an integral surface, containing a given curve C described by the parametric equations of the form

$$x=x_0(t),\ y=y_0(t),\ u=u_0(t).$$

where t is a parameter, Then the particular solution must be satisfy.

$$\left.\begin{aligned}\varphi(x_0(t), y_0(t), u_0(t))=c_1\\ \psi(x_0(t), y_0(t), u_0(t))=c_2\end{aligned}\right\}$$

Thus we have two relation, from which we can eliminate the parameter t to obtain a relation of the type $F(c_1, c_2)=0$ which leads to the solution of the give pdE.

III Geometric significant of Cauchy problem

Geometrically, there exists a surface $u = u(x,y)$ which passes through the curve Γ, whose parametric equations are

$$x = x_0(t),\ y = y_0(t),\ u = u_0(t).$$

and at every point of which the direction $(p, q, -1)$ of the normal is such that

$$F(x, y, u, p, q) = 0.$$

Problems:

(1) Find the general Soln of the first-order linear partial differential equation —

$$xu_x + yu_y = u$$

Soln The characteristic curves of this eqn are the solutions of the characteristic equation

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{u}$$

From $\frac{dx}{x} = \frac{dy}{y}$

$$\Rightarrow y\,dx = x\,dy$$
$$\Rightarrow y\,dx - x\,dy = 0$$
$$\Rightarrow \frac{y\,dx - x\,dy}{x^2} = 0$$
$$\Rightarrow d\left(\frac{y}{x}\right) = 0$$
$$\Rightarrow \frac{y}{x} = c_1$$
$$\Rightarrow \varphi(x, y, u) = \frac{y}{x} = c_1$$

Again

$$\frac{dy}{y} = \frac{du}{u}$$
$$\Rightarrow \log y = \log u + \log c_2$$
$$\Rightarrow \log\left(\frac{y}{u}\right) = \log c_2$$
$$\Rightarrow \frac{y}{u} = c_2$$
$$\Rightarrow \psi(x, y, u) = \frac{y}{u} = c_2$$

where c_1, c_2 are arbitrary constant.

∴ Thus the integral surface $\left.\begin{array}{l}\varphi = \frac{y}{x} = c_1 \\ \psi = \frac{y}{u} = c_2\end{array}\right\}$

∴ The general Soln $f(\varphi, \psi) = 0$

$$\Rightarrow f\left(\frac{y}{x}, \frac{y}{u}\right) = 0 \text{ where}$$

f is an arbitrary function.

Again $\frac{y}{x} = c_1$, $\frac{x}{u} = c_2 \Rightarrow \frac{u}{y} = c_3 \;\left(c_3 = \frac{1}{c_2}\right)$

Then $\frac{u}{y} = f\left(\frac{y}{x}\right)$

$\therefore u(x,y) = y\, f\left(\frac{y}{x}\right)$,

Where f is an arbitrary function.

② Obtain the general solⁿ of the linear Euler equation $x u_x + y u_y = n u$.

⇒ The integral surface are the solⁿ of the characteristic eqⁿ $\frac{dx}{x} = \frac{dy}{y} = \frac{du}{nu}$.

Then the characteristic curves are

$$\phi(x,y,u) = \frac{y}{x} = c_1$$
$$\psi(x,y,u) = \frac{u}{x^n} = c_2$$

where c_1 and c_2 are arbitrary constant.

Hence, the general solⁿ is $f\left(\frac{y}{x}, \frac{u}{x^n}\right) = 0$

or $u(x,y) = x^n g\left(\frac{y}{x}\right)$

this shows that the solⁿ $u(x,y)$ is homogeneous function of x and y of degree n.

③ Find the general Sol^n of the linear eq^n $x^2u_x + y^2u_y = (x+y)u$. (H.W)

Ans → $f\left(\frac{xy}{u}, \frac{x-y}{u}\right)=0$

④ Show that the general Sol^n of the linear eq^n $(y-z)u_x + (z-x)u_y + (x-y)u_z = 0$ is
$u(x,y,z) = f(x+y+z,\ x^2+y^2+z^2)$. (H.W)

⑤ Find the solution of the equation
$u(x+y)u_x + u(x-y)u_y = x^2+y^2$, with the Cauchy data $u=0$ on $y=2x$.

Sol^n the characteristic eq^n are

$$\frac{dx}{u(x+y)} = \frac{dy}{u(x-y)} = \frac{du}{x^2+y^2} = \frac{y\,dx + x\,dy - u\,du}{0} = \frac{x\,dx - y\,dy - u\,du}{0}$$

consequently,

$\therefore\ y\,dx + x\,dy - u\,du = 0$
$\Rightarrow d(xy) - d\left(\frac{u^2}{2}\right) = 0$
$\Rightarrow d\left(xy - \frac{u^2}{2}\right) = 0$
$\Rightarrow xy - \frac{u^2}{2} = c_1$
$\Rightarrow 2xy - u^2 = c_1$

$x\,dx - y\,dy - u\,du = 0$
$\Rightarrow d\left(\frac{1}{2}(x^2-y^2-u^2)\right) = 0$
$\Rightarrow x^2 - y^2 - u^2 = c_2$

where c_1 and c_2 are constants. Hence, the general solution is

$$f(x^2-y^2-u^2,\ 2xy-u^2) = 0$$ when f is arbitrary function.

$\therefore$ using cauchy data: $u=0$ on $y=2x$.

$\therefore\ 2\cdot x\cdot 2x = c_1$
$\Rightarrow 4x^2 = c_1$

$x^2 - 4x^2 = c_2$
$\therefore\ 3x^2 = c_2$
$\therefore\ x^2 = \frac{c_2}{3}$

$\therefore\ 4\cdot\frac{c_2}{3} = c_1$
$\therefore\ 4c_2 = 3c_1$
$\Rightarrow 4(x^2-y^2-u^2) = 3(2xy-u^2)$

$\Rightarrow 7u^2 = 6xy + 4(x^2 + y^2)$

Obtain the sol^n of the linear equation

$$u_x - u_y = 1$$

with cauchy data $u(x,0) = x^2$

$\Rightarrow$ The characteristic eq^n are

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{du}{1}$$

then general sol^n

$$u - x = f(x+y)$$

using cauchy data.

$$x^2 - x = f(x)$$

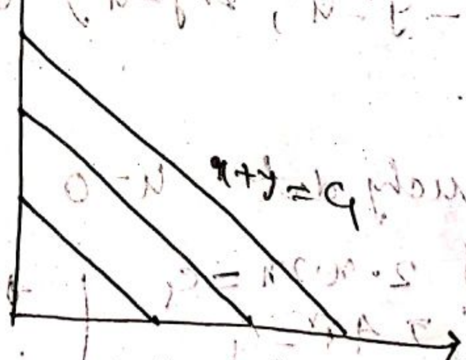
Then $u = x + f(x+y)$

$$= x + (x+y)^2 - \cancel{x} - y$$

$$= (x+y)^2 - y$$

$$\therefore u(x,y) = (x+y)^2 - y$$

Geometrically ~~$x+y = c_1$ are drawn~~ in



The value of u must be given at one point on each characteristic ~~with~~ which intersects the line $y=0$ only at one point; as show in the figure.