Type 1 based on Rule I for solving

$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ which is the characteristic eqn of the PDE $P(x,y,u)p + Q(x,y,u)q = R(x,y,u)$

where $p = \frac{\partial u}{\partial x} = u_x$, $q = \frac{\partial u}{\partial y} = u_y$.

Suppose that one of the variables is either or cancels out from any two fractions of characteristic eqn, Then an integral can be obtained by the usual methods.

⇒ ① $\frac{y^2 u}{x} p + xuq = y^2$

Soln Characteristic eqn

$$\frac{dx}{\frac{y^2 u}{x}} = \frac{dy}{xu} = \frac{du}{y^2}$$

From first two fractions

$$\frac{dx}{\frac{y^2 u}{x}} = \frac{dy}{xu}$$

$\Rightarrow x^2 dx = y^2 dy$

$\Rightarrow \frac{x^3}{3} = \frac{y^3}{3} + \frac{c_1}{3}$

$\Rightarrow x^3 - y^3 = c_1 = \phi$

From last ~~two~~ an first fractions

$$\frac{dx}{\frac{y^2 u}{x}} = \frac{du}{y^2}$$

$\Rightarrow x\,dx = u\,du$

$\Rightarrow \frac{x^2}{2} = \frac{u^2}{2} + \frac{c_2}{2}$

$\Rightarrow u^2 - x^2 = c_2 = \psi$

∴ Then general soln

$f(x^3 - y^3, u^2 - x^2) = 0$.

② Solve $p\tan x + q\tan y = \tan u$

⇒ Characteristic eqn.

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{du}{\tan u}$$

Taking first two.

$\cot x\,dx - \cot y\,dy = 0$

$\Rightarrow \log\sin x - \log\sin y = \log c_1$

$\Rightarrow \frac{\sin x}{\sin y} = c_1$

Taking last two.

$\frac{\sin y}{\sin u} = c_2$

∴ Then the general Soln

$f\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin u}\right) = 0$

Q. Solve $(x^2+2y^2)u_x - xyu_y = xu$.

$\Rightarrow$ The characteristic eqn —

$$\frac{dx}{x^2+2y^2} = \frac{dy}{-xy} = \frac{du}{xu}$$

$\therefore$ Taking $\dfrac{dy}{-xy} = \dfrac{du}{xu}$

$$\Rightarrow \frac{dy}{-y} = \frac{du}{u}$$

$$\Rightarrow -\log y = \log u - \log c_1$$

$$\Rightarrow \log uy = \log c_1$$

$$\Rightarrow uy = c_1$$

Again $\dfrac{dx}{x^2+2y^2} = \dfrac{dy}{-xy}$

$$\therefore \frac{dx}{dy} = \frac{x^2+2y^2}{-xy}$$

$$\therefore 2x\frac{dx}{dy} = \frac{2x^2+4y^2}{-y}$$

$$\therefore 2x\frac{dx}{dy} + \left(\frac{2}{y}\right)x^2 = -4y$$

$\therefore$ Putting $x^2 = v$.

$$\therefore 2x\frac{dx}{dy} = \frac{dv}{dy}$$

$$\therefore \frac{dv}{dy} + \left(\frac{2}{y}\right)v = -4y$$

$$I.F = e^{\int \frac{2}{y}dy} = e^{2\log y} = y^2$$

$$\therefore yv^2 = \int\{(-4y)xy^2\}dy + c_2$$

$$\therefore y^2x^2 + y^4 = c_2$$

$\therefore$ General soln $f(uy,\ y^2(x^2+y^2)) = 0$

Type 2 based on Rule II:

(1) $u_x + 3u_y = 5u + \tan(y-3x)$

$\Rightarrow$ characteristic eqn

$$\frac{dx}{1} = \frac{dy}{3} = \frac{du}{5u+\tan(y-3x)}$$

Taking $\frac{dx}{1} = \frac{dy}{3}$

$\Rightarrow 3dx = dy$

$\Rightarrow 3x - y = -c_1$

$\Rightarrow y - 3x = c_1$

Taking $\frac{dx}{1} = \frac{du}{5u+\tan(y-3x)}$

$\Rightarrow \frac{dx}{1} = \frac{du}{5u+\tan c_1}$

$\Rightarrow x = \frac{1}{5}\log(5u+\tan c_1) + \frac{1}{5}c_2$

$\Rightarrow 5x - \log(5u+\tan c_1) = c_2$

$\therefore$ Then General soln

$$f(y-3x,\ 5x - \log(5u+\tan(y-3x))) = 0$$

or $5x - \log(5u+\tan(y-3x)) = f(y-3x)$.

(2) Solve $u_x + 3u_y = u + \cot(y-3x)$. (HW)

(3) Solve $(u-2y^2)u_x = (u - yu_y)(u - y^2 - 2x^3)$.

Type-3: (multipliers method)

we know if $\frac{A}{B} = \frac{C}{D} = \frac{E}{F}$ ~~then~~ ~~where~~

and P_1, Q_1, R_1 be some function or constant then

$$\frac{A}{B} = \frac{C}{D} = \frac{E}{F} = \frac{P_1 A + Q_1 C + R_1 E}{P_1 B + Q_1 D + R_1 F}$$

This P_1, Q_1, R_1 are called multipliers.

Q) Solve $(mu - ny)u_x + (nx - lu)u_y = ly - mx$

Sol^n: $\frac{dx}{mu-ny} = \frac{dy}{nx-lu} = \frac{du}{ly-mx} \rightarrow (1)$

choosing x, y, u as multipliers, each fraction

(1) $= \frac{x\,dx + y\,dy + u\,du}{x(mu-ny) + y(nx-lu) + u(ly-mx)}$

$= \frac{x\,dx + y\,dy + u\,du}{0}$

$\Rightarrow x\,dx + y\,dy + u\,du = 0$

$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} + \frac{u^2}{2} = \frac{c_1}{2}$

$\Rightarrow x^2 + y^2 + u^2 = c_1$

Again, choosing l, m, n as multipliers each fraction of (1)

$= \frac{l\,dx + m\,dy + n\,du}{l(mu-ny) + m(nx-lu) + n(ly-mx)}$

$= \frac{l\,dx + m\,dy + n\,du}{0}$

$\Rightarrow lx + my + nu = c_2$

Then general Sol^n

$f(x^2+y^2+u^2, lx+my+nu) = 0$

Scanned by CamScanner

(11) Find the general solution of the linear equation

$$x^2 u_x + y^2 u_y = (x+y)u.$$

$\Rightarrow$ The characteristic equation associated with the given eqn

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{du}{(x+y)u} \longrightarrow (1)$$

From the first two of these equations, we find

$$x^{-1} - y^{-1} = c_1 \longrightarrow (2)$$

where c_1 is an arbitrary constant. ~~we get~~ then chose 1, -1, 0 as multipliers

(1) is equal to $= \dfrac{dx - dy}{x^2 - y^2}$

Then $\dfrac{dx-dy}{x^2-y^2} = \dfrac{du}{(x+y)u}$

$$\Rightarrow \frac{dx-dy}{x-y} = \frac{du}{u}$$

This gives $\Rightarrow \log(x-y) = \log u + \log c_2$

$$\Rightarrow \log\frac{(x-y)}{u} = \log c_2$$

$$\Rightarrow \frac{x-y}{u} = c_2 \longrightarrow (3)$$

Then general soln $f\left(\frac{1}{x} - \frac{1}{y}, \frac{x-y}{u}\right) = 0$

Furthermore, (1) and (3)

$$\frac{y-x}{xy} = c_1 \Rightarrow \frac{-uc_2}{xy} = c_1$$

$$\Rightarrow \frac{xy}{u} = c_3$$

$\therefore$ Then another ~~general~~ form of the solution is $f\left(\frac{xy}{u}, \frac{x-y}{u}\right) = 0$

Existence and Uniqueness of Integral Surface passing through a given curve

⇒ Given PDE $P(x,y,u)\,p + Q(x,y,u)\,q = R(x,y,u)$ with initial curve —— (1)

$$\Gamma : x_0(t),\ y_0(t),\ u_0(t).$$

The number of solutions of this PDE are according as

Unique solution:

if $\dfrac{P(x_0, y_0, u_0)}{\frac{dx_0}{dt}} \neq \dfrac{Q(x_0, y_0, u_0)}{\frac{dy_0}{dt}}$

or $\Delta = \begin{vmatrix} P(x_0(t), y_0(t), u_0(t)) & Q(x_0(t), y_0(t), u_0(t)) \\ \frac{dx_0(t)}{dt} & \frac{dy_0(t)}{dt} \end{vmatrix} \neq 0$

then given PDE has unique Sol^n.

NO Solution:

If $\dfrac{P(x_0, y_0, u_0)}{\frac{dx_0}{dt}} = \dfrac{Q(x_0, y_0, u_0)}{\frac{dy_0}{dt}} \neq \dfrac{R(x_0, y_0, u_0)}{\frac{du_0}{dt}}$

then (1) has no solution.

Infinite Sol^n: If $\dfrac{P(x_0, y_0, u_0)}{\frac{dx_0}{dt}} = \dfrac{Q(x_0, y_0, u_0)}{\frac{dy_0}{dt}} = \dfrac{R(x_0, y_0, u_0)}{\frac{du_0}{dt}}$

then (1) has infinite solution.

Example: consider the cauchy's problem $u_x - u_y = 2$. Check whether equation has unique solution passing through the curve $(2t, t, 2t)$.

Example: consider the cauchy's problem $u_x = u$, $u(x,0) = \sin x$. Find the number of sol^n passing through the curve $(s, 0, \sin s)$