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(1) Canonical Form of First-order Linear PDE:

It is often convenient to transform the more general first-order linear partial differential eqn $a(x,y)u_x + b(x,y)u_y + c(x,y)u = d(x,y)$. → (1)
into a canonical (or standard) form which can be easily integrated to find the general soln of (1). We use the characteristics of this equation (1) to introduce the new transformation by equation
$\xi = \xi(x,y)$, $\eta = \eta(x,y)$. where ξ and η are once continuously differentiable and their Jacobian $J(x,y) = \xi_x \eta_y - \xi_y \eta_x$ is nonzero in a domain of interest so that x and y can be determined uniquely from the system of equation (1)

Thus by chain rule,

$$\left.\begin{aligned} u_x &= u_\xi \xi_x + u_\eta \eta_x \\ u_y &= u_\xi \xi_y + u_\eta \eta_y \end{aligned}\right\} \rightarrow (2)$$

we substitute these partial derivatives (2) into (1) to obtain the equation

$$A u_\xi + B u_\eta + cu = d.$$

where $A = a\xi_x + b\xi_y$, $B = a\eta_x + b\eta_y$ → (3)

From (3) we see that $B=0$ if η is a soln of the first-order equation $a\eta_x + b\eta_y = 0$.

This eqn has infinitely many solns. We can obtain one of them by assigning initial condition on a non-characteristic initial curve and solving the resulting initial-value problem according to the method described earlier.

Note: Since $\eta(x,y)$ satisfies eqn $a\eta_x + b\eta_y = 0$ then $\eta(x,y) =$ constant are always characteristic curve of equation ①. Thus, one set of the new transformations are the characteristic curve of ①. The second set $\xi(x,y) =$ constant can be chosed to be any one parameter family of smooth curve which are nowhere tangent to the family of the characteristic curves.

If $A \neq 0$ in a nbd of some point in the domain D in which $\eta(x,y)$ is defined and $J \neq 0$...

For if $A = 0$ at some point of D, then $B = 0$ at the same point. Consequently, eqn ③ would form a system of linear homogeneous equation in a and b. Where the Jacobian J is the determinant of its coefficient matrix.

Since $B = 0$ and $A \neq 0$ in D, we can divided $Au_\xi + Bu_\eta + cu = d$ by A to obtain the canonical form.

$$u_\xi + \alpha(\xi,\eta)u = \beta(\xi,\eta) \Rightarrow ④$$

where $\alpha(\xi,\eta) = \frac{c}{A}$ $\beta(\xi,\eta) = \frac{d}{A}$

The eqn ④ represent an ordinary differential eqn with ξ as the independent variable and η as a parameter which may be treated as constant. This eqn ④ is called canonical form of eqn ① in term of (ξ,η).

Ex: Reduce each of the following equation $yu_x + u_y = x$ to canonical form, and obtain the general sol^n

$\Rightarrow$ comparing given eq^n with

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = d(x,y)$$

we get $a = y$ $b = 1$, $c = 0$, $d = x$.

The characteristic eq^{n}s are

$$\frac{dx}{y} = \frac{dy}{1} = \frac{du}{x}.$$

$\therefore \frac{dx}{y} = \frac{dy}{1}$

$\therefore dx = y\,dy$

$\therefore x = \frac{y^2}{2} + c_1$

$\therefore x - \frac{y^2}{2} = c_1$.

Let $\xi(x,y) = x - \frac{y^2}{2} = c_1$

we chose $\eta(x,y) = y = c_2$.

consequently $u_x = u_\xi$ and $u_y = -y u_\xi + u_\eta$.

Hence the eq^n reduce to.

$$u_\eta = \xi + \frac{1}{2}\eta^2$$

$$\therefore u(\xi,\eta) = \xi\eta + \frac{1}{6}\eta^3 + f(\xi)$$

where f is arbitrary function.

$\therefore$ General sol^n $u(x,y) = xy - \frac{1}{3}y^3 + f\left(x - \frac{y^2}{2}\right)$

VII Method of Separation of Variables

Ex: Solve the initial-value problem

$$u_x + 2u_y = 0 \quad \rightarrow (1), \quad u(0,y) = 4e^{-2y}.$$

Let $u(x,y) = X(x)Y(y) \neq 0$ is a soln.

$$u_x = X'Y$$
$$u_y = XY'$$

Now $u_x + 2u_y = 0$

$$\Rightarrow X'Y + 2XY' = 0$$
$$\Rightarrow X'Y = -2XY'$$
$$\Rightarrow \frac{X'(x)}{2X(x)} = -\frac{Y'(y)}{Y(y)} \rightarrow (2)$$

Since the left-hand side of this eqn is a function of x only and the right-hand is a function of y only, it follows that (2) can be true if both sides are equal to the same constant value λ which is called an arbitrary separation constant.

$$\therefore \frac{X'(x)}{2X(x)} = -\frac{Y'(y)}{Y(y)} = \lambda$$

$\therefore X'(x) - 2\lambda X(x) = 0$ | $Y'(y) + \lambda Y(y) = 0$

$\therefore X(x) = Ae^{2\lambda x}$ | $Y(y) = Be^{-\lambda y}$.

Then

$$\therefore u(x,y) = Ae^{2\lambda x}Be^{-\lambda y}$$
$$= ABe^{\lambda(2x-y)}$$
$$= Ce^{\lambda(2x-y)}$$

where A, B and C are arbitrary constant $C = AB$

$$\therefore u(0,y) = 4e^{-2y}$$
$$\Rightarrow Ce^{-\lambda y} = 4e^{-2y}$$
$$\Rightarrow C = 4, \ \lambda = 2$$

$\therefore$ Final Soln

$$u(x,y) = 4e^{4x-2y}$$

Ex: Solve the eqn

$$y^2 u_x^2 + x^2 u_y^2 = (xyu)^2$$

Ex: use $u(x,y) = f(x) + g(y)$ to solve the eqn

$$u_x^2 + u_y + x^2 = 0$$

Ex: use $v = \ln u$ and $v = f(x) + g(y)$ to solve the eqn

$$x^2 u_x^2 + y^2 u_y^2 = u^2$$