Mathematical models

Classical Equations:

Partial differential equation arise frequenty in formulating fundamental laws of nature and in the study a wide variety of physical, chemical and biological models. we start with a special type of second-order linear partial differential equation for the following reasons.

(1) Second-order linear equation arise more frequently in a wide variety of applications.

(2) Their mathematical treatment is simpler and and easier to understand than that of first-order eqn in general.

usually, in almost all physical phenomena (or physical processes), the dependent variable $u = u(x, y, z, t)$ is a function of three space variables x, y, z and time variable t.

The three basic types of second-order partial differential equations are–

(a) The wave equation–

$$u_{tt} - c^2(u_{xx} + u_{yy} + u_{zz}) = 0.$$

(b) The heat equation

$$u_t - K(u_{xx} + u_{yy} + u_{zz}) = 0$$

(c) The Laplac equation

$$u_{xx} + u_{yy} + u_{zz} = 0.$$

In this section, we list a few more common linear partial differential equations of importance in applied mathematics, mathematical physics, and engineering science. such a list naturally cannot ever be complete. Included are only equations of most common interest;

(d) The poisson equation

$$\nabla^2 u = f(x, y, z)$$

(e) The Helmholtz equation

$$\nabla^2 u + \lambda u = 0$$

(f) The biharmonic equation

$$\nabla^4 u = \nabla^2(\nabla^2 u) = 0.$$

(g) The biharmonic wave eqn:

$$u_{tt} + c^2 \nabla^4 u = 0$$

(h) The telegraph equation

$$u_{tt} + a u_t + b u = c^2 u_{xx}.$$

(i) For a compressible fluid flow, Euler's Equation

$$u_t + (u \cdot \nabla) u = -\frac{1}{\rho} \nabla p.$$

$$\rho_t + \operatorname{div}(\rho u) = 0.$$

Where $u = (u, v, w)$ is the fluid velocity vector, ρ is the fluid density, and $p = p(\rho)$ is the pressure that relates p and ρ (the constitutive equation or eqn of state).

We will begin our study of these equations by first examining in detail the mathematical models representing physical problem.

The vibrating String

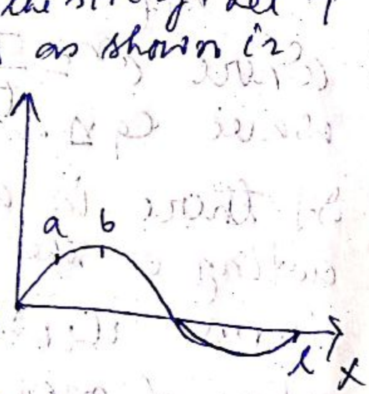
One of the most importent problem in mathematical physics is the vibrating ~~string~~ of a stretched string. Simplicity and frequent occurrence in many branches of mathematical physics make it a classic example in the theory of partial differential eqn.

Let us consider a stretched string of length l fixed at the end points. The problem here is to determine the equation of motion which characterizes the position $u(x,t)$ of the string at time t after an initial disturbance is given.

In order to obtain a simple equation, we make the following assumptions:

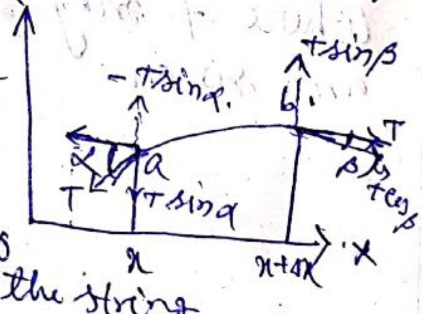
1. The string is flexible and elastic, i.e, the string cannot resist bending moment and thus the tension in the string is always in the direction of the tangent to the existing profile of the string.
2. There is no elongation of a single segment of the string and hence, by Hooke's law, the tension is constant.
3. The weight of the string is small compared with the tension in the string.
4. The deflection is small compared with the length of the string.
5. The slope of the displaced string at any point is small compared with unity.
6. There is only pure transverse vibration.

We consider a differential element of the string. Let T be the tension at the end points as shown in Figure. The forces acting on the element of the string in vertical direction $- T\sin\beta - T\sin\alpha$.



By Newton's second law of motion, the resultant force is equal to the mass times the acceleration. Hence

$$T\sin\beta - T\sin\alpha = \rho\,\delta s\, u_{tt} \rightarrow (1)$$



where ρ is the line density and δs is the smaller arc length of the string.

Since the slope of the displaced string is small, we have. $\delta s \simeq \delta x$

Since the angles α and β are small

$\sin\alpha \simeq \tan\alpha$, $\sin\beta \simeq \tan\beta$

Thus eqn (1) becomes

$$\tan\beta - \tan\alpha = \frac{\rho\,\delta x}{T} u_{tt} \rightarrow (2)$$

But, from calculus we know that $\tan\alpha$ and $\tan\beta$ are the slopes of the string at x and $x+\delta x$:

$\tan\alpha = u_x(x,t)$

and $\tan\beta = u_x(x+\delta x, t)$. at time t.

Equation (2) may thus be written as

$$\frac{1}{\delta x}\left[(u_x)_{x+\delta x} - (u_x)_x\right] = \frac{\rho}{T} u_{tt}$$

$$\frac{1}{\delta x}\left[u_x(x+\delta x,t) - u_x(x,t)\right] = \frac{\rho}{T} u_{tt}$$

In the limit as δx approaches zero, we find.

$$u_{tt} = c^2 u_{xx}.$$

In the limit as δx approaches zero, we find.

$$u_{tt} = c^2 u_{xx} \rightarrow (3)$$

where $c^2 = \frac{T}{\rho}$. This is called the one-dimensional wave eqn.

If there is an external force f per unit length acting on the string. Equation (3) assumes the form $u_{tt} = c^2 u_{xx} + F$, $F = \frac{f}{\rho}$.

where f may be pressure, gravitation, resistance and so on.