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Derivation of the one-dimentional heat Equation

In the early 1800, J. Fourier began a mathematical study of heat. A deeper understanding of heat flow had significant application in sicience and within industry.

A basis version of Fourier's efforts in the problem

$$\boxed{u_t = c^2 u_{xx}}$$

with BC's $u(0,t) = 0 = u(l,t)$

$u(x,0) = f(x)$.

where $u(x,t)$ is the temperature at position x at time t. c^2 is a constant and f is a given function.

Derivation:

To derive this one-dimentional heat equation we follow some empirical law's

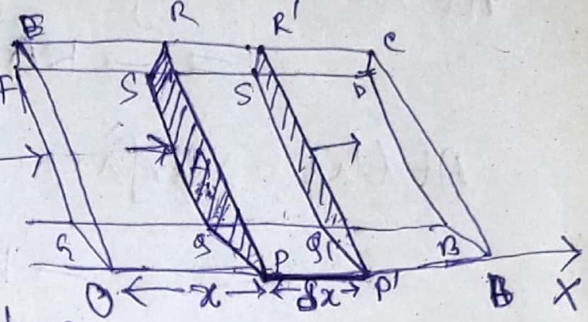
1. Heat flow from a higher temprerature to a lower temperature.

2. The amount of heat in a body is proportional to its mass and temperature ($H = m \times s \times t$)

3. The rate of heat flow across on area is proportional to the area and to the temparature gradient normal to the area.

wher the constant of proportional K is called the ~~terminal~~ thermal conductivity.

consider a homogeneous rectangular bar, with cross section area 'A'
Density $= \rho$
The specific heat of this meatal bar is $= s$
Thermal conductivity $= K$



Let $u(x,t)$ be the teamperature at any paint P.
conside the element of the bar between the place PQRS and P'Q'R'S' ~~at a distance~~ δx. distance from the origin with thikness δx. distance from the origin is x.
Then distance of the surface PQSR from the origin is x and the surface P'Q'R'S' $(x+\delta x)$.

Let the temparatur at the surface PQSR is ~~δu~~ $\delta u + u(x,t)$ and at the surface P'Q'S'R' is $u + \delta u$.

Then chance in temparature $= u + \delta u - u$
in a slab thikness of δx. $= \delta u$.

The mass of the element $= \rho \times A \delta x$ $\left[D = \frac{m}{V}.\ m = DV\right]$
$= A\rho\delta x$.

Now the quantity of heat stroed in the slabe element $=$ (mass x specific heat x change in temparatur
$= A\rho\delta x \times s \times \delta u$
$= A\rho s\delta x\,\delta u$

So the rate of increase of heat in the slabe element $(R) = A\rho s\delta x \frac{\delta u}{\delta t}$

Now ~~rate of~~ rate of heat flow from Fourier's law —

At the surface PQSR is

$$= -KA\left[\frac{\partial u(x,t)}{\partial x}\right]_x$$

At the surface P'Q'SR'

$$= -KA\left[\frac{\partial u}{\partial x}\right]_{x+\delta x}.$$

The rate of heat flow in the space

$$= -KA\left[\frac{\partial u}{\partial x}\right]_x + KA\left[\frac{\partial u}{\partial x}\right]_{x+\delta x}.$$

$$\therefore \; A\rho\,\delta x \cdot \frac{\delta u}{\delta t} = -KA\left[\frac{\partial u}{\partial x}\right]_x + KA\left[\frac{\partial u}{\partial x}\right]_{x+\delta x}$$

$$\Rightarrow A\rho\,\delta x\,\frac{\delta u}{\delta t} = KA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right].$$

$$\Rightarrow \frac{\delta u}{\delta t} = \frac{KA}{A\rho}\cdot\frac{1}{\delta x}\left[u_x(x+\delta x,t) - u_x(x,t)\right]$$

$$\Rightarrow \lim_{\delta t\to 0}\frac{\delta u}{\delta t} = \frac{KA}{A\rho}\lim_{\delta x\to 0}\frac{u_x(x+\delta x,t) - u_x(x,t)}{\delta x}$$

$$\Rightarrow \frac{\partial u}{\partial t} = \frac{KA}{A\rho}\frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow \frac{\partial u}{\partial t} = \underbrace{\frac{K\cancel{A}}{\cancel{A}\rho}}_{c^2}\frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow \frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow \boxed{u_t = c^2 u_{xx}}$$ where $c^2 = \frac{K}{\rho}$