Date - 15-09-20

Classification of Second order Linear Equation

① Second-order Equations in Two Independent Variables:

The general linear ~~eq~~ second-order PDE in one dependent variable u ~~and two independent variable x and y~~ may be written as

$$\sum_{i,j=1}^{n} A_{ij} u_{x_i x_j} + \sum_{i=1}^{n} B_i u_{x_i} + Fu = G \longrightarrow ①$$

$\left[\begin{array}{l} x, y \\ u_{xy} = u_{yx} \end{array}\right.$

in which we assume $A_{ij} = A_{ji}$ and A_{ij}, B_i, F, and G are real-valued functions defined in some region of space $(x_1, x_2, \ldots, x_n)$. $\boxed{A_{ij} = A_{ji}}$

Here we shall be concerned with second-order eqn in dependent variable u and the independent variables x and y. Hence eqn ① can be put in the form

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

$$\Rightarrow \boxed{Rr + Ss + Tt + Pp + Qq + Eu = G}$$

where the coefficients are functions of x and y and do not vanish simultaneously.

where $r = u_{xx} = \frac{\partial^2 u}{\partial x^2}$

$s = u_{xy} = \frac{\partial^2 u}{\partial x \partial y}$

$t = u_{yy} = \frac{\partial^2 u}{\partial y^2}$

$p = \frac{\partial u}{\partial x}$

$q = \frac{\partial u}{\partial y}$

We shall assume that the function u and the coefficients are twice continuously differentiable in some domain in $\mathbb{R}^2$

III Classification of 2-nd order PDE

The classification of PDE is suggested by the classification of the quadratic eqn of conic section in analytic geometry.

The eqn of 2-nd order PDE

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G.$$

represent hyperbola if $B^2 - 4AC > 0$

parabola if $B^2 - 4AC = 0$

ellipse if $B^2 - 4AC < 0$.

After find the type of the eqn, a transformation can be always be found to reduce the given equation to a canonical form in a given domain.

To transform to a canonical form we make a change of idependent variables. Let the new variables be —

$$\xi = \xi(x,y), \quad \eta = \eta(x,y).$$

Assuming that ξ and η are twice continuously differentiable and that the Jacobian

$$J = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0.$$

in the region under consideration, then x and y can be determined uniquely from the new eqn.

Let x and y be twice continuously diffn function of ξ and η. Then

$$u_x = u_\xi \xi_x + u_\eta \eta_x \qquad u_y = u_\xi \xi_y + u_\eta \eta_y.$$

$$u_{xx} = u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}.$$

$$u_{xy} = u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_{\eta\eta}\eta_x\eta_y + u_\xi\xi_{xy} + u_\eta\eta_{xy}$$

$$u_{yy} = u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi\xi_{yy} + u_\eta\eta_{yy}.$$

Substituting these values we get

$$A^* u_{\xi\xi} + B^* u_{\xi\eta} + C^* u_{\eta\eta} + D^* u_\xi + E^* u_\eta + F^* u = G^*$$

where $A^* = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2$

$B^* = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$

$C^* =$

(H.W)

$D^* =$

$E^* =$

$F^* =$

$G^* =$

Note: ① If we transform the given PDE to a new PDE through this transformation the nature of the eqn remains invariant under such transformation.

② The classification of eqn ① depends on the coefficient $A(x,y)$, $B(x,y)$, and $C(x,y)$ at given point (x,y)

we shall, therefore rewrite eqn as

$$A u_{xx} + B u_{xy} + C u_{yy} = H(x, y, u, u_x, u_y) \quad \text{②}$$

and eqn as

$$A^* u_{\xi\xi} + B^* u_{\xi\eta} + C^* u_{\eta\eta} = H^*$$

Canonical forms.

We suppose first the none of A, B, C is zero. Let ξ, η be new variables s.t the coeff 2 A^*, C^* in the eqn (1) vanish.

Then $A^* = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0$

$C^* = A\eta_x^2 + B\eta_x\eta_y + C\eta_x^2 = 0$

thes two eqn are say type and hence we may write this in the form

$$A\alpha_x^2 + B\alpha_x\alpha_y + C\alpha_y^2 = 0$$

In which α stand for either of the function ξ or η.

$$A\left(\frac{\alpha_x}{\alpha_y}\right)^2 + B\left(\frac{\alpha_x}{\alpha_y}\right) + C = 0.$$

Along $\alpha =$ costant.

$d\alpha = \alpha_x dx + \alpha_y dy = 0$

$\therefore \alpha_x dx + \alpha_y dy = 0.$

$$\therefore \boxed{\frac{dy}{dx} = -\frac{\alpha_x}{\alpha_y}}$$

Then $A\left(\frac{dy}{dx}\right)^2 + B\left(\frac{dy}{dx}\right)^2 + C = 0.$

$\Rightarrow A\lambda^2 + B\lambda + C = 0$ when $\lambda = \frac{dy}{dx}$

$\Rightarrow \lambda = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$

$\Rightarrow \frac{dy}{dx} = \frac{-B + \sqrt{B^2 - 4AC}}{2A}$

$\frac{dy}{dx} = \frac{-B - \sqrt{B^2 - 4AC}}{2A}$

Scanned by CamScanner

Thus eqn, which are know as the characteristic eqn which are ode, of family of curves in the xy-plane along whic ξ = constn

The integrals of η = cont. the eqn's are called characteristic curve.

Since the eqn are 1st-order ODE the soln may be written as

$$\varphi_1(x,y) = c_1$$

$$\varphi_2(x,y) = c_2$$

Hence the transformations

$$\xi = \varphi_1(x,y),\ \eta = \varphi_2(x,y)$$

<u>Prob</u>: Find the cannonical form of

$$u_{xx} + x^2 u_{yy} = 0.$$

<u>Soln</u> comparing the given eqn

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

$\therefore$

$A = 1$

$B = 0$

$C = x^2$

$D = 0$

$E = 0$

$F = 0$

$G = 0$

$\therefore$ Now $B^2 - 4AC = 0 - 4 \cdot 1 \cdot x^2$

$= -4x^2 < 0 \quad \forall x \neq 0$

$= 0$ along y-axis

given eqn is elliptic over all $x \in \mathbb{R}, x \neq 0$
parabola for $x = 0$.

For $x \neq 0$ i.e. when eqn is elliptic we caculate characteristic eqn.

$$A\left(\frac{dy}{dx}\right)^2 + B\left(\frac{dy}{dx}\right) + C = 0$$

$$\Rightarrow A\lambda^2 + B\lambda + C = 0$$

$$\Rightarrow \lambda^2 + 0\lambda + x^2 = 0$$

$$\Rightarrow \lambda^2 = -x^2$$

$$\Rightarrow \lambda = \pm ix$$

$\therefore \frac{dy}{dx} = ix$

$\Rightarrow dy = ix\,dx$

$\Rightarrow y = \frac{ix^2}{2} + \frac{c_1}{2}$

$\Rightarrow 2y - ix^2 = c_1$

$\frac{dy}{dx} = -ix$

$\therefore dy = -ix\,dx$

$\therefore y = -\frac{ix^2}{2} + \frac{c_2}{2}$

$\therefore 2y + ix^2 = c_2$

$$\therefore \boxed{\begin{aligned} \xi &= 2y - ix^2 \\ \eta &= 2y + ix^2 \end{aligned}}$$

$\therefore \xi + \eta = 4y$

$\therefore y = \frac{\xi + \eta}{4} \Rightarrow 2y = \frac{1}{2}(\xi + \eta) = \alpha$

$-2ix^2 = \xi - \eta$.

$\therefore x^2 = \frac{1}{-2i}(\xi - \eta) = \beta$.

$u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi \cdot (-2ix) + u_\eta \cdot (2ix)$.

$u_{xx} = \frac{\partial}{\partial x}(u_\xi \cdot (-2ix)) + \frac{\partial}{\partial x}(u_\eta (2ix))$

$= u_{\xi\xi} \frac{\partial \xi}{\partial x} \cdot (-2ix) + (-2i) u_\xi$
$+ u_{\eta\eta} \frac{\partial \eta}{\partial x} \cdot 2ix + u_\eta \cdot 2i$

$= -4x^2 u_{\xi\xi} - 2i u_\xi + 4x^2 u_{\eta\eta}$
$+ 2i u_\eta$.

$\therefore u_{xx} = -4x^2 u_{\xi\xi} + 4x^2 u_{\eta\eta} - 2i u_\xi + 2i u_\eta$

$u_{yy} =$

An: $\boxed{u_{\alpha\alpha} + u_{\beta\beta} = -\frac{1}{2\beta} u_\beta}$

HW

Q) Find the canonical form of

(a) $x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = 0$.

(b) $y^2 u_{xx} - x^2 u_{yy} = 0$

Q)