$\ldots 3u_{yy} + u_y = 0$

Prob:

Use $u = f(\xi)$, $\xi = \dfrac{x}{\sqrt{4kt}}$ to solve the heat eqn $u_t = k u_{xx}$, $-\infty < x < \infty$, $t > 0$, with B'cs, $u(x,0) = \begin{cases} 0 & x < 0 \\ u_0 & x > 0 \end{cases}$

where k and u_0 are constant.

$\Rightarrow$ The given PDE

$$u_t = k u_{xx} \Rightarrow k u_{xx} - u_t = 0.$$

Comparing the given eqn

$$A u_{xx} + B u_{xt} + C u_{tt} + D u_x + E u_t + F = G.$$

we get $A = K$, $B = 0$, $C = 0$

$D = 0$ $E = -1$, $F = 0$ $G = 0$.

$\therefore$ $B^2 - 4AC = 0 - 4 \cdot K \cdot 0 = 0$.

$\therefore$ Then the given heat eqn is parabol.

where $\lambda = \dfrac{dt}{dx}$

$\therefore$ Now $A\lambda^2 - B\lambda + C = 0$

$\Rightarrow K\lambda^2 = 0$

$\Rightarrow \lambda^2 = 0$

$\Rightarrow \lambda = 0$

$\therefore \dfrac{dt}{dx} = 0 \Rightarrow dt = 0 \Rightarrow t = c_1$

So we take $\boxed{\xi = t}$

Let $\boxed{\eta = x}$ s.t $J(\xi, \eta) = \begin{vmatrix} \xi_x & \xi_t \\ \eta_x & \eta_t \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0$

So the characteristic curve are

$\xi = t \quad \eta = x$

$$u_t = \frac{\partial u(\xi,\eta)}{\partial t} = \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial t} = u_\xi \cdot 1 + u_\eta \cdot 0 = u_\xi$$

$$u_x = \frac{\partial u(\xi,\eta)}{\partial x} = \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = u_\xi \cdot 0 + u_\eta \cdot 1.$$

$\therefore u_x = u_\eta.$

$$\therefore u_{xx} = \frac{\partial u_\eta(\xi,\eta)}{\partial x} = \frac{\partial u_\eta}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial u_\eta}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} = u_{\eta\eta}$$

$\therefore \quad u_t - k u_{xx} = 0$

$\Rightarrow \boxed{u_\xi - k u_{\eta\eta} = 0}$ So it is same as previous PDE, therefor we thikn for new transformation

Let $u(x,t) = f(\xi)$

where $\xi = \frac{x}{\sqrt{4kt}}$

$$\therefore \quad u_t = \frac{\partial}{\partial t} f(\xi) = f'(\xi)\frac{\partial \xi}{\partial t} = f'(\xi)\left(-\frac{1}{2}\frac{x}{\sqrt{4kt^3}}\right) = -\frac{1}{2}\frac{x}{\sqrt{4kt^3}} f'(\xi)$$

$$u_x = \frac{\partial f(\xi)}{\partial x} = f'(\xi)\frac{\partial \xi}{\partial x} = f'(\xi)\frac{1}{\sqrt{4kt}}$$

$$u_{xx} = \frac{\partial}{\partial x}\left(f'(\xi)\cdot\frac{1}{\sqrt{4kt}}\right) = f''(\xi)\frac{1}{\sqrt{4kt}}\cdot\frac{\partial \xi}{\partial x}$$
$$= f''(\xi)\left(\frac{1}{\sqrt{4kt}}\right)^2 = \frac{f''(\xi)}{4kt}$$

$\therefore$ Now from the given eqn.

$u_t - k u_{xx} = 0$

$$\Rightarrow -\frac{1}{2}\frac{x}{\sqrt{4kt^3}} f'(\xi) - \cancel{k}\cdot\frac{f''(\xi)}{4kt} = 0$$

$$\Rightarrow f''(\xi) + 2\xi f'(\xi) = 0.$$

$\therefore f''(\xi) + 2\xi f'(\xi) = 0$

$\therefore$ ~~$\int f''(\xi)\, d\xi$~~.

$\therefore f''(\xi) = -2\xi f'(\xi)$

$\therefore \dfrac{f''(\xi)}{f'(\xi)} = -2\xi$. (Interation both side)

$\therefore \log f'(\xi) = -\dfrac{2\xi^2}{2} + \log A$ where A is constant

$\therefore \log \dfrac{f'(\xi)}{A} = -\xi^2$

$\therefore \dfrac{f'(\xi)}{A} = e^{-\xi^2}$

$\therefore f'(\xi) = A e^{-\xi^2}$

$\therefore f(\xi) = A\int_0^{\xi} e^{-\alpha^2} d\alpha + B$ where B is arbitary constant

$\therefore u(x,t) = A\int_0^{\xi} e^{-\alpha^2} d\alpha + B$.

$\therefore$ Initialy $u(x,0) = 0$ $x < 0$.

$\Rightarrow A\int_0^{-\infty} e^{-\alpha^2} d\alpha + B = 0 \rightarrow (1)$

$u(x,0) = u_0$ $x > 0$

$A\int_0^{\infty} e^{-\alpha^2} d\alpha + B = u_0 \rightarrow (2)$

$\therefore$ From (2) $-$ (1) $A\int_0^{\infty} e^{-\alpha^2} d\alpha + \cancel{B} - A\int_0^{-\infty} e^{-\alpha^2} d\alpha - \cancel{B} = u_0$

$\Rightarrow A\left[\int_0^{\infty} e^{-\alpha^2} d\alpha - \int_0^{-\infty} e^{-\alpha^2} d\alpha\right] = u_0$

$\Rightarrow A\left[\int_0^{\infty} e^{-\alpha^2} d\alpha + \int_{-\infty}^{0} e^{-\alpha^2} d\alpha\right] = u_0$.

$\Rightarrow A\int_{-\infty}^{\infty} e^{-\alpha^2} d\alpha = u_0$

$\left[\begin{array}{l} \text{Let } \phi(\alpha) = e^{-\alpha^2} \\ \phi(-\alpha) = e^{-\alpha^2} \\ \phi(\alpha) = \phi(-\alpha) \\ \text{so } e^{-\alpha^2} \text{ is even func}^n \end{array}\right.$

$\Rightarrow A \cdot 2\int_0^{\infty} e^{-\alpha^2} d\alpha = u_0$.

$\therefore 2A\int_0^\infty e^{-\alpha^2}\,d\alpha = u_0$

$\therefore$ ~~2A Γ(α+~~

$2A\cdot\frac{1}{2}\sqrt{\pi} = u_0$

$\therefore A = \frac{u_0}{\sqrt{\pi}}$

$\therefore$ from (1)

$$A\int_0^\infty e^{-\alpha^2}\,d\alpha + B = u_0 \quad \left[\because \text{from Gamma function } \Gamma(n)=\int_0^\infty e^{-x}x^{n-1}\,dx,\ n>0\right.$$

$$\therefore \frac{u_0}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2} + B = u_0$$

$$\therefore B = u_0 - \frac{u_0}{2} = \frac{u_0}{2}.$$

$\therefore$ Then the solution is.

$$u(x,t) = A\int_0^\xi e^{-\alpha^2}\,d\alpha + B$$

$$= \frac{u_0}{\sqrt{\pi}}\int_0^\xi e^{-\alpha^2}\,d\alpha + \frac{u_0}{2}.$$

$$\therefore \boxed{u(x,t) = u_0\left[\frac{1}{\sqrt{\pi}}\int_0^{\frac{x}{\sqrt{4kt}}} e^{-\alpha^2}\,d\alpha + \frac{1}{2}\right]}$$

Prob: Let $u(x,t)$ satisfy the IVP

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty,\ t>0$$

with BC's $u(x,0) = \begin{cases} 0 & x<0 \\ 1 & x>0 \end{cases}$

Then find the value of $\lim_{t\to 0^+} u(1,t)$.

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