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The cauchy problem and Wave Equations

On the theory of ordinary differential equations, by the initial-value problem we mean the problem of finding the Solution of a given differential equation with the appropriate number of initial conditions prescribed at an initial point. For example, the second-order ordinary differential equation

$$\frac{d^2u}{dt^2} = f\left(t, u, \frac{du}{dt}\right)$$

[redacted]

and the initial conditions

$$u(t_0) = \alpha, \left(\frac{du}{dt}\right)(t_0) = \beta.$$

Constitute an initial-value problem.

An analogous problem can be defined in the case of partial differential equations. Here we shall state the problem involving second-order partial differential equations in two independent variables. We consider a second-order partial differential equation for the function u in the independent variables x and y and suppose that this equation can be solved explicitly for u_{yy}, and hence, can be represented in the from

$$u_{yy} = f(x, y, u, u_x, u_y, u_{xx}, u_{xy}) \rightarrow (1)$$

For some value $y = y_0$, we prescribe the initial values of the unknown function and of the derivative with respect to y, $u(x, y_0) = f(x)$, $u_y(x, y_0) = g(x)$. $\rightarrow (2)$

The problem of determining the solution of equation (1) satisfying the initial conditions (2) is known as the initial-value-problem. For example, the initial-value problem of a ~~vari~~ vibrating string is the problem of finding the solution of the wave equation

$$u_{tt} = c^2 u_{xx}$$

Satisfying the initial conditions

$$u(x, t_0) = u_0(x), \quad u_t(x, t_0) = v_0(x).$$

Where $u_0(x)$ is the initial displacement and $v_0(x)$ is the initial velocity.

In the initial-value problems, the initial values usually refer to the data assigned at $y = y_0$. It is not essential that these values be given along the line $y = y_0$; they may very well be prescribed along some curve L_0 in the xy plane. In such a context, the problem is called the cauchy problem instead of initial-value problem. although the two names are actually synonymous.

VII we consider the Euler equation

$$Au_{xx} + Bu_{xy} + Cu_{yy} = F(x, y, u, u_x, u_y) \rightarrow (3)$$

where A, B, C are functions of x and y. Let (x_0, y_0) denote points on a smooth curve L_0 in the xy plane. Also let the parametric equations of this curve L_0 be $x_0 = x_0(\lambda)$, $y_0 = y_0(\lambda) \rightarrow (4)$ where λ is a parameter.

we suppose that two functions $f(\lambda)$ and $g(\lambda)$ are prescribed along the curve L_0. The cauchy problem is now one of determining the solution $u(x, y)$ of equation (3) in the neighborhood of the curve L_0 satisfying the cauchy conditions

$$\left.\begin{aligned} u &= f(\lambda) \quad (i) \\ \frac{\partial u}{\partial n} &= g(\lambda) \quad (ii) \end{aligned}\right\} \rightarrow (5)$$

on the curve L_0 where n is the direction of the normal to L_0 which lies to the left of L_0 in the counterclockwise direction of increasing arc length. The function $f(\lambda)$ and $g(\lambda)$ are called cauchy data.

VIII For every point on L_0, the value of u is specified by equation (5). Thus, the curve L_0 represented by equation (4) with the condition (5) yields a twisted curve L in (x, y, u) space whose projection on the xy plane is the curve L_0. Thus, the solution of the cauchy problem is a surface called an integral surface, in the (x, y, u) space passing through L and satisfying the condition (5) (ii) which represent a tangent plane to the integral surface along L.

If the function $f(\lambda)$ is differentiable, then along the curve L_0, we have

$$\frac{du}{d\lambda} = \frac{\partial u}{\partial x}\frac{dx}{d\lambda} + \frac{\partial u}{\partial y}\frac{dy}{d\lambda} = \frac{df}{d\lambda} \rightarrow (6)$$

and
$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial x}\frac{dx}{dn} + \frac{\partial u}{\partial y}\frac{dy}{dn} = g \rightarrow (7)$$

but
$$\frac{dx}{dn} = -\frac{dy}{d\lambda} \quad \text{and} \quad \frac{dy}{dn} = \frac{dx}{d\lambda} \rightarrow (8)$$

The eqn (7) may be written as

$$\frac{\partial u}{\partial n} = -\frac{\partial u}{\partial x}\frac{dy}{d\lambda} + \frac{\partial u}{\partial y}\frac{dx}{d\lambda} = g. \rightarrow (9)$$

Since
$$\begin{vmatrix} \frac{dx}{d\lambda} & \frac{dy}{d\lambda} \\ -\frac{dy}{d\lambda} & \frac{dx}{d\lambda} \end{vmatrix} = \frac{dx}{d\lambda}\cdot\frac{dx}{d\lambda} + \frac{dy}{d\lambda}\cdot\frac{dy}{d\lambda} \neq 0 \rightarrow (10)$$

It is possible to find u_x and u_y on L_0 from the system of equation (6) and 9.
Since u_x and u_y are known on L_0, we find the higher derivatives by first differentiating u_x and u_y with respect to λ. Thus, we have

$$\frac{\partial^2 u}{\partial x^2}\frac{dx}{d\lambda} + \frac{\partial^2 u}{\partial x \partial y}\frac{du}{d\lambda} = \frac{d}{d\lambda}\left(\frac{\partial u}{\partial x}\right) \rightarrow (11)$$

$$\frac{\partial^2 u}{\partial x \partial y}\frac{dx}{d\lambda} + \frac{\partial^2 u}{\partial y^2}\frac{dy}{d\lambda} = \frac{d}{d\lambda}\left(\frac{\partial u}{\partial y}\right) \rightarrow (12)$$

From eqn (3)

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} = F$$

where F is known since u_x and u_y have been found. The system of equations can be solved by u_{xx}, u_{xy} and u_{yy} if

$$\begin{vmatrix} \frac{dx}{d\lambda} & \frac{dy}{d\lambda} & 0 \\ 0 & \frac{dx}{d\lambda} & \frac{dy}{d\lambda} \\ A & B & C \end{vmatrix} \neq 0$$

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The eqn. $A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0$

is called the characteristic equation. It is then evident that the necessary condition for obtaning the second derivatives is that the curve L_0 must not be a characteristic curve.