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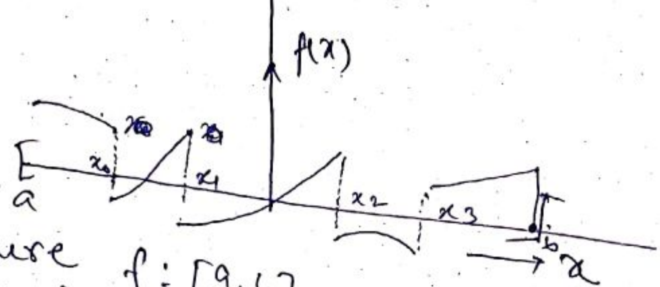
Fourier Series and Integral with Applications:

Piecewise continuous function and Periodic Function:

A single-valued function f is said to be piecewise continuous in a interval $[a, b]$ if there exists finitely many point $a = x_1 < x_2 < \cdots < x_n = b$, such that f is continuous in the interval $x_j < x < x_{j+1}$ and the one-sided limits $f(x_j+)$ and $f(x_{j+1}-)$ exists for all $j = 1, 2, 3, \ldots, n-1$.



From the figure $f : [a, b] \to \mathbb{R}$ be a piecewise continuous function, which is continuous at $[a, x_0]$, $[x_0, x_1]$, (x_1, x_2), (x_2, x_3), (x_3, b).

Let $f(x) = \frac{1}{x}$, $x \in [0, 1]$, here f is not piecewise continuous, since $\lim_{x \to 0+} f(x)$ does not exists.

Property:

(i) If f is piecewise continuous in an interval $[a, b]$ then it is bounded and integrable on $[a, b]$.

(2) product of two piecewise continuous function is piecewise continuous on a common interval.

(3) if f is piecewise continuous in $[a, b)$ and f' is continuous on $x_j < x < x_{j+1}$ $\forall j = 1, 2, \ldots, n-1$ and limits $f'(x_j+)$ and $f'(x_j-)$ exists, then f is said to be piecewise smooth

* if f'' is continuous in each $[x_j, x_{j+1}]$ and $f''(x_j+)$ $f''(x_j-)$ exists then f is piecewise very smooth

VIII Periodic: A piecewise continuous function $f(x)$ in an interval $[a,b]$ is said to be periodic if there exists a real positive number p such that $f(x+p) = f(x)$. for all x, p is called period of f.

EX! let $f(x) = \sin x$, $f(x) = \cos x$ } this are periodic function

i.e. $f(x+2\pi) = \sin(x+2\pi) = \sin x$.

$\therefore$ $\sin x$ is periodic with period 2π.

property: ~~f(x)~~ let f is periodic function with period p then ——

(i) $f(x+2p) = f(x+p+p) = f(x+p) = f(x)$

(ii) $f(x+3p) = f(x+2p+p) = f(x+2p) = f(x)$

(iv) $f(x+np) = f(x)$ $\forall n \in \mathbb{Z}$

(v) let $f_1, f_2, \dots f_k$ have the period p and c_k arbitrary ~~posit~~ constant then

$f = c_1 f_1 + c_2 f_2 + \dots + c_k f_k$ has the period p.

(vi) A constant function is also a periodic function with arbitrary period p.

EX! let $f(x) = 5$ $f(x+\pi) = f(x)$.

IX Systems of orthogonal Functions:

A sequence of function $\{\varphi_n(x)\}$ is said to be orthogonal with respect to the weight function $q(x)$ on the interval $[a,b]$ if

$$\int_a^b \varphi_m(x)\varphi_n(x) q(x)\,dx = 0 \quad m \neq n.$$

if $m = n$ the we have

$$\|\varphi_n\| = \left[\int_a^b \varphi_n^2(x)\, q(x)\,dx\right]^{1/2}$$

which is called the norm of the orthogonal system $\{\varphi_n(x)\}$.

EX! The sequence of function $\{\sin mx\}$, $m = 1, 2, \dots$ form an orthogonal system on the interval $[-\pi, \pi]$

because $$\int_{-\pi}^{\pi} \sin mx \sin nx\,dx = \begin{cases} 0 & m \neq n \\ & m = n \end{cases}$$

$q(x) = 1$ (taken)

An orthogonal system $\{\varphi_1, \varphi_2, \cdots, \varphi_n\}$ where n may be finite or infinite which satisfies the relation

$$\int_a^b \varphi_m(x)\varphi_n(x)\, q(x)\, dx = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

is called an orthonormal system of function on $[a,b]$.

Ⓘ <u>Fourier Series:</u> The function $1, \cos x, \sin x, \cdots \cos 2x, \sin 2x, \cdots$ are mutually orthogonal to each other in the interval $[-\pi, \pi]$ and are linearly independent. Thus, we formally associate a trigonometric series with any piecewise continuous periodic function $f(x)$ of period 2π and write

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx),$$

where the symbol $\sim$ indicates an association of a_0, a_k, and b_k to f in some unique manner.

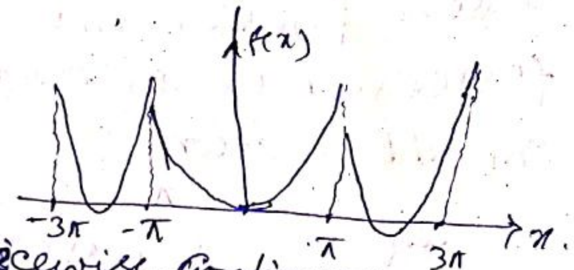
Ⓘ If the series is convergent then the coefficient —

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos kx\, dx$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin kx\, dx.$$

① Find the fourier series expansion for the function $f(x) = x + x^2 \quad -\pi < x < \pi$.



clearly this function is piecewise continuous.

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\,dx = \frac{2\pi^2}{3}$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\cos kx\,dx$$

$$= \frac{1}{\pi}\left[\frac{x\sin kx}{k}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{\sin kx}{k}\,dx\right]$$

$$+ \frac{1}{\pi}\left[\frac{x^2\sin kx}{k}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{2x\sin kx}{k}\,dx\right].$$

$$= -\frac{2}{k\pi}\left[-\frac{x\cos kx}{k}\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{2x\sin kx}{k}\,dx\right]$$

$$= \frac{4}{k^2}\cos k\pi = \frac{4}{k^2}(-1)^k$$

$$b_k = \frac{1}{\pi}\int f(x)\sin kx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin kx\,dx$$

$$= -\frac{2}{k}\cos k\pi = -\frac{2}{k}(-1)^k \quad k = 1, 2, \ldots$$

$$\therefore f(x) = \frac{2\pi^2}{2\times 3} + \sum_{k=1}^{\infty}\left[\frac{4}{k^2}(-1)^k\cos kx - \frac{2}{k}(-1)^k\sin kx\right].$$

$$= \frac{\pi^2}{3} - 4\cos x + 2\sin x + \cos 2x - \sin 2x$$

Let $f(x)$ be an even function define on the interval $[-\pi,\pi]$. Since $\cos kx$ is an even function and $\sin kx$ an odd function $f(x)\cos kx$ is an even function and $f(x)\sin nx$ an odd function.

$$\therefore a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos kx \quad k=0,1,--$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx = 0.$$

Then Fourier series of an

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$$

* If $f(x)$ is odd function and $f(x)\cos kx$ is an odd and $f(x)\sin kx$ is an even.

Then $a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx = 0 \quad k=0,1,--$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin kx\,dx$$

$\therefore$ Then fourier series of $f(x) = \sum_{k=1}^{\infty} b_k \sin kx$.

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