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Method of separation of variables

Introduction:

The method of separation of variables combined with the principal of superposition is widely used to solve initial boundary-value problems involving linear partial differential equations. usually, the dependent variable $u(x,y)$ is expressed in the separable form $u(x,y) = X(x)\,Y(y)$ where X and Y are function of x and y respectively. In many cases, the partial differential equation reduces to two ordinary differential equations for X and Y. A similar treatment can be applied to equations ~~of separability of a~~ in three or more independent variable.

This method of solution is also known as the Fourier method (or the method of eigenfunction expansion). Thus, the procedure outlined above leads to the important ideas of eigenvalues eigenfunctions and orthogonality, all of which are very general and powerfull for dealing with linear problem.

Separation of Variables:

we consider the second-order homogeneous partial differential equation

$$a^* u_{x^*x^*} + b^* u_{x^*y^*} + C^* u_{y^*y^*} + d^* u_{x^*} + e^* u_{y^*} + f^* u = 0 \quad \longrightarrow (1)$$

where a^*, b^*, c^*, d^*, e^* and f^* are function of x^* and y^*.

using canonical form concept, we can transfr this PDE to

$$a u_{xx} + c u_{yy} + d u_x + e u_y + f u = 0$$

using $x = x(x^*, y^*)$ $y = y(x^*, y^*)$.

where $\dfrac{\partial(x,y)}{\partial(x^*,y^*)} \neq 0$.

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This eqn is ~~~~ hyperbolic, parabolic or elliptic according as

$b^2 - 4ac > 0 \Leftrightarrow a = -c$

$b^2 - 4ac = 0 \Leftrightarrow a = 0 = c$.

$b^2 - 4ac < 0 \Leftrightarrow a = c$.

We assume a separable solution of (2) in the form.

$$u(x,y) = X(x)\,Y(y) \neq 0. \rightarrow (3)$$

where X and Y are, respectively, function of x and y alone, and are twice continuously differentiable. Substituting equations (3) in (2) we get

$$\boxed{a X''Y + cXY'' + dX'Y + eXY' + fXY = 0}$$

we write this eqn such as.

$$a_1(x)\, X''Y + b_1(y)\, XY'' + a_2(x) X'Y + b_2(y) XY' + [a_3(x) + b_3(y)] XY = 0.$$

$\therefore$ Dividing by XY we get

$$\therefore \left[a_1 \frac{X''}{X} + a_2 \frac{X'}{X} + a_3\right] + \left[b_1 \frac{Y''}{Y} + b_2 \frac{Y'}{Y} + b_3\right] = 0$$

$$\Rightarrow a_1 \frac{X''}{X} + a_2 \frac{X'}{X} + a_3 = -\left[b_1 \frac{Y''}{Y} + b_2 \frac{Y'}{Y} + b_3\right].$$

The left side of eqn is a function of x only and the right side of eqn depend only upon y. So

$$a_1 \frac{X''}{X} + a_2 \frac{X'}{X} + a_3 = \lambda$$

$$-\left[b_1 \frac{Y''}{Y} + b_2 \frac{Y'}{Y} + b_3\right] = \lambda.$$

where λ is constant.

Then we get two ODE.

$$\left.\begin{aligned} a_1 X'' + a_2 X' + (a_3 - \lambda) X = 0 \\ b_1 Y'' + b_2 Y' + (b_3 + \lambda) Y = 0. \end{aligned}\right\}$$

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Thus $u(x,y)$ is the soln of the eqn if $X(x)$ and $Y(y)$ are the soln of the ordinary differential equations respect.

Note! If the coefficient of PDE are constant then canonical form does not necessary.

IV Types of boundary value problem:

(i) if u is prescribed on a boundary then it is called Dirichlet condition and problem is called Dirichlet problem.

(ii) $\frac{\partial u}{\partial n}$ is prescribed on a boundary then it is called Neumann condition.

(iii) If $\frac{\partial u}{\partial n} + hu$ is prescribed on a boundary where $\frac{\partial u}{\partial n}$ is the directional derivative of u along the outward normal to the boundary and h is a given continuous function on the boundary.

V Vibrating String problem

$$u_{tt} - c^2 u_{xx} = 0 \quad 0 < x < l \quad t > 0$$
$$u(x,0) = f(x) \quad 0 \le x \le l$$
$$u_t(x,0) = g(x) \quad 0 \le x \le l$$
$$u(0,t) = 0 \quad t \geq 0$$
$$u(l,t) = 0 \quad t \geq 0$$

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Sol$\underline{n}$: Let $u(x,t) = X(x)\,T(t)$.

$$u_t(x,t) = X(x)\,T'(t).$$

$$u_{tt}(x,t) = X(x)\,T''(t).$$

Now $u_x(x,t) = X'(x)\,T(t)$

$$u_{xx}(x,t) = X''(x)\,T(t).$$

∴ Then from given eq$\underline{n}$ —

$$X(x)\,T''(t) - c^2 X''(x)\,T(t) = 0.$$

$$\therefore\; X(x)\,T''(t) = c^2 X''(x)\,T(t)$$

$$\therefore\; \frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = \cancel{k} \quad (\text{a constant})$$

Now consider the following cases —

Case 1: $K = 0$.

$$\therefore\; \frac{\cancel{X}''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = 0.$$

$\therefore\; \cancel{X}''(t) = 0$ $\qquad$ $X''(x) = 0$.

$\therefore\; T'(t) = c_1$ $\qquad$ $\therefore X(x) = c_3 x + c_4$.

$T(t) = c_1 t + c_2$

$$\therefore\; u(x,t) = X(x)\,T(t) = (c_3 x + c_4)(c_1 t + c_2).$$

Now using boundary condition —

$u(0,t) = 0 \Rightarrow (c_3 \cdot 0 + c_4)(c_1 t + c_2) = 0$

$\Rightarrow \cancel{c_4 c_1 t + c_2 = 0}$

$\Rightarrow c_4(c_1 t + c_2) = 0.$

~~$u(l,t) = 0$~~ $\Rightarrow$ $\Rightarrow c_4 = 0$.

$u(l,t) = 0$.

$\Rightarrow (c_3 l + 0)(c_1 t + c_2) = 0$

$\Rightarrow c_3 l = 0$

$\Rightarrow c_3 = 0$.

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For $k=0$ we get $c_3 = c_4 = 0 \Rightarrow X(x) = 0$.
which gives $u(x,t) = 0$. i.e. zero sol^n.

Case 2: $k > 0$, Let $k = \alpha^2$ (α is constant).

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{c^2 T(t)} = \alpha^2$$

$\therefore \frac{X''(x)}{X(x)} = \alpha^2$

$\therefore X''(x) = \alpha^2 X(x)$

$\therefore X''(x) - \alpha^2 X(x) = 0$

$\therefore (D^2 - \alpha^2) X = 0$

$\therefore D = \pm \alpha$

$\therefore X(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$

$T''(t) = \alpha^2 c^2 T(t)$

$\therefore T''(t) - \alpha^2 c^2 T(t) = 0$

$\therefore (D^2 - \alpha^2 c^2) T = 0$

$\therefore D^2 = \alpha^2 c^2$

$\therefore D = \pm \alpha c$

$\therefore T(t) = c_3 e^{\alpha c t} + c_4 e^{-\alpha c t}$

$\therefore u(x,t) = (c_1 e^{\alpha x} + c_2 e^{-\alpha x})(c_3 e^{\alpha c t} + c_4 e^{-\alpha c t})$.

Again $u(0,t) = 0 \Rightarrow (c_1 + c_2)(c_3 e^{\alpha c t} + c_4 e^{-\alpha c t}) = 0$

$\Rightarrow c_1 + c_2 = 0$ — (a) [$\because c_3 e^{\alpha c t} + c_4 e^{-\alpha c t} \neq 0$ since if this is zero we get zero sol^n]

$u(l,t) = 0 \Rightarrow (c_1 e^{l\alpha} + c_2 e^{-\alpha l})(c_3 e^{\alpha c t} + c_4 e^{-\alpha c t}) = 0$

$\Rightarrow c_1 e^{l\alpha} + c_2 e^{-l\alpha} = 0$. (b)

For (a) and (b)

$$\begin{vmatrix} 1 & 1 \\ e^{l\alpha} & e^{-l\alpha} \end{vmatrix} = e^{-l\alpha} - e^{l\alpha} \neq 0.$$

So $c_1 + c_2 = 0$ and $c_1 e^{l\alpha} + c_2 e^{-l\alpha} = 0$
has only zero sol^n.

$\therefore c_1 = c_2 = 0. \Rightarrow X(x) = 0$

$\therefore u(x,t) = 0$.
which gives zero sol^n

Case III): Let $K < 0$ i.e. $K = -\lambda^2$

Then $\frac{X''(x)}{X(x)} = \frac{T''(t)}{c^2 T(t)} = -\lambda^2$

$\Rightarrow X''(x) = -\lambda^2 X(x)$

$\Rightarrow X''(x) + \lambda^2 X(x) = 0$

$\Rightarrow (D^2 + \lambda^2) X = 0$

$\therefore D = \pm i\lambda$

$\therefore X(x) = c_1 \cos\lambda x + c_2 \sin\lambda x$

$T''(t) = -\lambda^2 c^2 T(t)$

$T''(t) + \lambda^2 c^2 T(t) = 0.$

$\therefore (D^2 + \lambda^2 c^2) T(t) = 0.$

$D = \pm i\lambda c.$

$\therefore T(t) = c_3 \cos\lambda ct + c_4 \sin\lambda ct$

$\therefore u(x,t) = (c_1 \cos\lambda x + c_2 \sin\lambda x)(c_3 \cos\lambda ct + c_4 \sin\lambda ct).$

$u(0,t) = 0.$

$\Rightarrow (c_1 \cos\lambda 0 + c_2 \sin\lambda (0))(c_3 \cos\lambda ct + c_4 \sin\lambda ct) = 0.$

$\Rightarrow c_1 = 0.$

$\therefore u(x,t) = (c_2 \sin\lambda x)(c_3 \cos\lambda ct + c_4 \sin\lambda ct).$

Again $u(l,t) = 0$

$\Rightarrow c_2 \sin\lambda l\,(c_3 \cos\lambda ct + c_4 \sin\lambda ct) = 0.$

$\Rightarrow c_2 \sin\lambda l = 0.$

$\Rightarrow \sin\lambda l = 0$ [Since $c_2 \neq 0$, if $c_2 = 0$ we get trivial sol^n].

$\Rightarrow \sin\lambda l = \sin n\pi$ $n = 0, \pm 1, \pm 2 \,--$

$\Rightarrow \lambda l = n\pi$

$\Rightarrow \boxed{\lambda = \frac{n\pi}{l}}$

$\therefore u_n(x,t) = c_2 \sin\frac{n\pi x}{l}\left(c_3 \cos\frac{n\pi ct}{l} + c_4 \sin\frac{n\pi ct}{l}\right).$

$= c_2 c_3 \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right) + c_2 c_4 \sin\frac{n\pi ct}{l} \cdot \sin\frac{n\pi x}{l}$

$u_n(x,t) = a_n \sin\frac{n\pi x}{l} \cos\frac{n\pi ct}{l} + b_n \sin\frac{n\pi ct}{l} \sin\frac{n\pi x}{l}$ $n = 0, \pm 1, \pm 2$

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Now $u_n(x,t) = a_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi ct}{l}\right)$.

So using Superposition $n=0,\pm1,\pm2$ —

condition —

$$u(x,t) = \sum_{n=0}^{\infty} \left(a_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right) + b_n \sin\frac{n\pi x}{l} \sin\frac{n\pi ct}{l}\right)$$

Now $u(x,0) = f(x)$.

$$\Rightarrow \sum_{n\geq 0}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right) = f(x).$$

$$\Rightarrow a_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\, dx.$$

$$u_t(x,t) = \sum_{n\geq 0}^{\infty} (-)a_n \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi ct}{l}\right) \times \left(\frac{n\pi c}{l}\right) + b_n \sin\frac{n\pi x}{l} \cos\frac{n\pi ct}{l} \left(\frac{n\pi c}{l}\right)$$

$$= \sum_{n\geq 0}^{\infty} \left(\frac{n\pi c}{l}\right)\left\{b_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right) - a_n \sin\frac{n\pi x}{l}\sin\left(\frac{n\pi ct}{l}\right)\right\}$$

$u_t(x,0) = g(x)$.

$$\Rightarrow \sum_{n\geq 0}^{\infty} \frac{n\pi c}{l} \cdot b_n \sin\frac{n\pi x}{l} = g(x).$$

$$\Rightarrow b_n = \frac{2}{l}\int_0^l g(x) \frac{n\pi c}{l} \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2n\pi c}{l^2}\int_0^l g(x) \sin\left(\frac{n\pi x}{l}\right) dx.$$

$\therefore$ Then the general sol^n —

$$u(x,t) = \sum_{n\geq 0}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi ct}{l}\right)$$

when $a_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\, dx$, $b_n = \frac{2n\pi c}{l^2}\int_0^l g(x) \sin\frac{n\pi x}{l}\, dx$

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